CONVERGENCE OF THE BIRKHOFF SPECTRUM FOR NONINTEGRABLE OBSERVABLES

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Abstract. We consider interval maps with countably many full branches and observables with polynomial tails. We show that the Birkhoff spectrum is real analytic and that its convergence to the Hausdorff dimension of the repeller is governed by the polynomial tail exponent. This result extends previous work by Arima on more regular observables and demonstrates how the tail behaviour influences the structure of the Birkhoff spectrum. Our proof relies on techniques from thermodynamic formalism and tail estimates for the observable and our applications are to natural observations on Gauss maps, Lüroth transformations as well as to a the first return time for a class of induced Manneville-Pomeau maps.

1. Introduction

Averaging observables along an orbit is a basic procedure in ergodic theory. The pointwise ergodic theorem ensures that, with respect to any ergodic measure, these averages converge to a constant almost everywhere. However, it does not provide information about the exceptional set of measure zero. For large classes of dynamical systems $T : X \to X$ and functions $g : X \to \mathbb{R}$, the decomposition induced by its Birkhoff averages exhibits considerable complexity. Indeed, for $\alpha \in \mathbb{R}$ the associated level set is defined by

$$J(\alpha) = \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(T^i x) = \alpha \right\}.$$

This induces a decomposition of X given by

$$X = \bigcup_{\alpha\in\mathbb{R}} J(\alpha) \cup J_{ir},$$

where J_{ir} is the set of points for which the Birkhoff average fails to converge. In many situations every level set $J(\alpha)$ is dense in X. When X is a metric space, the structure of this decomposition can be analyzed via the function $\alpha \mapsto b(\alpha) := \dim_H J(\alpha)$, where $\dim_H$ denotes the Hausdorff dimension. The function $b(\cdot)$ is referred to as the *Birkhoff spectrum*. The study of the Birkhoff spectrum began in the late 1990s in the context of hyperbolic conformal dynamical systems and Hölder functions (see [P]). In this setting, if g is not not cohomologous to a constant, the decomposition induced by the Birkhoff averages is extremely complicated. The set of attainable values α forms a nontrivial interval, and each level set $J(\alpha)$ is dense in X. Despite this complexity, the function b is remarkably regular. Indeed, it is real analytic. This regularity is a consequence of the well-behaved thermodynamic formalism associated with the system.

We consider dynamical systems given by interval maps $F : [0, 1] \to [0, 1]$ with countably many (full) branches. The dynamics is concentrated in the repeller $\Lambda \subset [0, 1]$, and we assume the existence of a

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measure of maximal dimension μ, see Section 2.2 for details. We also consider functions $\tau : \Lambda \to \mathbb{R}$, referred to as observables. These are assumed to satisfy conditions (see Section 2.3) which imply that their tail with respect to the measure μ has polynomial decay, meaning there exists $0 < \beta < 1$ such that $\mu(\tau > n) \asymp n^{-\beta}$. In this setting, we go on to prove that the function $b(\alpha)$ is real analytic and that $\lim_{\alpha\to\infty} b(\alpha) = \dim_H \Lambda$, see Proposition 3.7. Moreover, in analogy to recent results obtained by Arima [A], we prove the following,

Theorem 1.1. *Let $\Lambda \subset [0,1]$ be the repeller associated to the map F (as in Section 2.2) and let $\tau : \Lambda \to \mathbb{R}$ be an observable (as in Section 2.3), then*

$$\lim_{\alpha\to\infty} \left(dim_H \Lambda - dim_H J(\alpha)\right) \alpha^x = \begin{cases} 0 & \text{if } x > \frac{\beta}{1-\beta}; \\ \infty & \text{if } x < \frac{\beta}{1-\beta}. \end{cases} \tag{1.1}$$

In other words, the rate at which $b(\alpha)$ converges to $\dim_H \Lambda$ is controlled by a function of the polynomial exponent of the tail. This, in particular, implies that for observables as described in Section 2.3, different values of β lead to fundamentally distinct Birkhoff spectra.

As above, the observables we consider have (non-summable) polynomial tails with respect to the geometric measure as a consequence of more subtle (and technical) assumptions. These are hypothesis that allow us to control the growth and the interplay between the number of cylinders of length one for which the values of the observable lies in a certain fixed interval (Section 2.3 (H1)–(H3)) and the μ measure of the these cylinders (Section 2.3 (H2)). In Section 2.4, under somewhat stronger assumptions, we clarify the meaning and implications of (H1) (H2) and (H3).

A natural example for this theory is the induced map of the Manneville-Pomeau map together with the observable defined as the corresponding first return time map (see Example 5.1). In Section 5 we construct several examples that illustrate the assumptions on the observable and the system. Systems and observables that satisfy the hypotheses of our results can also occur as first return time maps for flows. This is readily seen simply by considering the suspension flow with base system F and roof function equal to τ. Other examples stem from the maps associated to certain numeration systems such as continued fractions expansions.

A result related to Theorem 1.1 was obtained by Arima [A] for similar dynamical systems, but for a class of more regular observables. In [A, Theorem 1.1] it is shown that the Birkhoff spectrum again converges to the Hausdorff dimension of the repeller at a polynomial rate. However, in that setting, the rate is determined by the Hausdorff dimension of the repeller, $\dim_H \Lambda$. Contrary to our Theorem, in order for that result to hold the dimension has to be strictly smaller than one.

It should be stressed that for this class of dynamical systems but for a different class of observables, the convergence of the Birkhoff spectra to the Hausdorff dimension of the repeller can be exponential [A, Proposition 1.2] (see also [IJ2]).

Our method of proof combines modifications of ideas developed by Arima in [A] together with tail estimates for the observable. All of these are thermodynamic in nature.

2. Preliminaries

In this section we introduce the objects we are going to study throughout the article, we also recall results and definitions that will be used and take the opportunity to fix notation. For two real sequences $(a_n)_n, (b_n)_n$, write $a_n = C^{\pm} b_n$ if there exists a uniform constant $C > 0$ such that for all $n \in \mathbb{N}$, $C^{-1} \leq \frac{a_n}{b_n} \leq C$; write $a_n \lesssim b_n$ if for all n large, $a_n \leq C b_n$, and $a_n \asymp b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$. The cardinality of a set E is denoted by $\#E$.

2.1. **The symbolic space and thermodynamic formalism.** Let (Σ, σ) be the one-sided full-shift shift over the countable alphabet $\mathcal{A} = \mathbb{N}$. That is, we consider the space $\Sigma := \{(x_n)_{n\in\mathbb{N}} : x_i \in \mathbb{N}\}$ together with the shift map $\sigma : \Sigma \to \Sigma$ defined by $\sigma(x_1x_2x_3\dots) = (x_2x_3\dots)$. Endow the space Σ with the topology generated by the cylinder sets,

$$[i_1 i_2 \dots i_n] := \{(x_n) \in \Sigma : x_i = i_j \text{ for } j \in \{1, 2, \dots n\}\}.$$

Note that with this topology the space Σ is non-compact. Moreover, the system has infinite entropy. We define the *n-th variation* of a function $\psi : \Sigma \to \mathbb{R}$ by

$$\mathrm{var}_n(\psi) = \sup_{(i_1\dots i_n)\in\mathbb{N}^n} \sup_{x,y\in[i_1i_2\dots i_n]} |\psi(x) - \psi(y)|.$$

A function $\psi : \Sigma \to \mathbb{R}$ is of *summable variations* if $\sum_{j=2}^{\infty} \mathrm{var}_j(\psi) < \infty$ and *locally Hölder* if there exists $M > 0$, $A_\psi \in (0, 1)$ such that $\mathrm{var}_j(\psi) \leq M A_\psi^j$ for all $j \geq 1$.

The thermodynamic formalism in this setting was studied by Mauldin and Urbański [MU2] and also by Sarig [S1]. We briefly recall the basic properties that will be used through the article. Let $\psi : \Sigma \to \mathbb{R}$ be a function of summable variations. The *topological pressure* of ψ is defined by

$$P_\sigma(\psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \psi(\sigma^i x)\right).$$

Denote by $\mathcal{M}_\sigma$ the space of $\sigma-$invariant probability measures. The Variational Principle [MU2, S1] states that

$$P_\sigma(\psi) = \sup\left\{h(\nu) + \int \psi \, d\nu : \nu \in \mathcal{M}_\sigma, \int \psi \, d\nu > -\infty\right\},$$

where $h(\nu)$ denotes the entropy of the measure ν. An invariant measure $\nu \in \mathcal{M}_\sigma$ is an *equilibrium state* for ψ if $P_\sigma(\psi) = h(\nu) + \int \psi \, d\nu$, whenever $P_\sigma(\psi) < \infty$.

An equivalent condition of the pressure being finite is given by the following result.

Lemma 2.1. [MU2, Lemma 2.1.9] *We have that* $P_\sigma(\psi) < \infty$ *if and only if*

$$\sum_{a\in\mathcal{A}} \exp\left(\sup_{x\in[a]} \psi(x)\right) < \infty.$$

In addition, it is possible to approximate the pressure defined on this non-compact space by the topological pressure of compact invariant subspaces.

Lemma 2.2. *(Approximation property,* [MU2, Theorem 2.1.6]*) If* $\psi : \Sigma \to \mathbb{R}$ *has summable variations then*

$$P_\sigma(\psi) = \sup\left\{h(\nu) + \int \psi \, d\nu : \nu \in \mathcal{M}_\sigma \text{ and compactly supported}\right\}.$$

The above result also means that if $K \subset \Sigma$ is a compact invariant subset and we denote by $P_K(\cdot)$ the restriction of the pressure to the set K, then $P_\sigma(\psi) = \lim_K P_K(\psi)$ where the limit is taken over larger and larger invariant compact subsets. There is a special class of measures that will be of interest to us.

Definition 2.3. *A measure* $\nu \in \mathcal{M}_\sigma$ *is said to have the* Gibbs property *if there exists* $M > 1$ *such that for all* $n \in \mathbb{N}$*, all* $n-$*cylinder* $[i_1, \dots, i_n]$ *and* $x \in [i_1, \dots, i_n]$*,*

$$\frac{1}{M} \leq \frac{\nu([i_1, \dots, i_n])}{\exp\left(S_n\psi(x) - nP(\psi)\right)} \leq M. \tag{2.1}$$

The following result summarises the properties of the pressure in this setting, see [MU2, S3] and more specifically [IJ1, Theorem 2.3].

Proposition 2.4. *Let (Σ, σ) be the full-shift and $\psi : \Sigma \to \mathbb{R}$ a non-positive function of summable variations. Then, there exists $t^* \geq 0$ such*

$$P_\sigma(t\psi) = \begin{cases} \infty & \text{if } t < t^*; \\ \text{finite and real analytic} & \text{if } t > t^*. \end{cases}$$

Moreover, if $t > t^*$ then there exists a unique equilibrium measure for $t\psi$ and it is a Gibbs measure.

Remark 2.5. *The original theorem in* [IJ1] *assumes ψ locally Hölder instead of summable variations; but according to the argument and the proof of* [S2, Theorem 4]*, the same statement holds for ψ of summable variations.*

When the pressure is real analytic there exists a formula for its first derivative. Indeed, if $t > t^*$ then

$$P'_\sigma(t\psi)\Big|_{t=t_0} = \int \psi \, d\mu_{t_0}, \tag{2.2}$$

where μ_{t_0} is the unique equilibrium measure for $t_0\psi$, see for example [MU2, Proposition 2.6.13].

2.2. **The class of maps.** Given a compact non degenerate interval $[a, b] \subset \mathbb{R}$, let $\{I_n\}_n \subset [a, b]$ be a countable collection closed subintervals such that their interiors are pairwise disjoint and let $F : \cup_n I_n \to [a, b]$ be a map. The *repeller* of the map F is defined by

$$\Lambda := \{x \in [a, b] : F^j(x) \in \cup_n I_n \text{ for all } j \in \mathbb{N}\}.$$

We make the following assumptions on the map F.

- (F1) The Hausdorff dimension of the boundary is zero, that is $\dim_H(\overline{\cup_n \partial I_n}) = 0$, where ∂I denotes the boundary of the set I.
- (F2) Expansiveness. The map is of class C^1 on $\text{int} I_n$, the interior of each interval I_n, for every $n \in \mathbb{N}$. Moreover, there exists $A > 1$ such that $|F'(x)| > A$ for every $x \in \cup_n \text{int} I_n$.
- (F3) The map F restricted to Λ is topologically conjugated to the full shift (Σ, σ) via the projection map $\pi : \Sigma \to \Lambda$.
- (F4) The function $|F'| \circ \pi$ is of summable variations and has finite $1-$variation.

Note that the Markov structure that we have assumed implies that there is a bijection between the space $\mathcal{M}_\sigma$ and that of F-invariant probability measures, that we denote by $\mathcal{M}_F$. For a continuous function $g : \Lambda \to \mathbb{R}$ we define the topological pressure of g with respect to F by

$$P_F(g) := \sup \left\{ h(\nu) + \int g \, d\nu : \nu \in \mathcal{M}_F, \int g \, d\nu > -\infty \right\}.$$

If $g \circ \pi$ is of summable variations we have that $P_F(g) = P_\sigma(g \circ \pi)$. Therefore, results obtained for one of the pressure are translatable to the other pressure. Hence, we identify them with the notation $P(\cdot)$.

Recall that for each $\nu \in \mathcal{M}_F$, its *Lyapunov exponent* is given by $\lambda(\nu) := \int \log |F'| \, d\nu$. We now state our final hypothesis on the map F, a Bowen-type formula.

- (F5) Assume that the function $-(\dim_H \Lambda) \log |F'|$ has zero pressure and an equilibrium state that we denote by μ.

Remark 2.6. *There exist examples of maps that satisfy conditions (F1)-(F4) but not (F5), see* [MU1, Example 5.3]. *We have that $\dim_H(\mu) = \dim_H \Lambda$ and hence it is called measure of maximal dimension. We stress that under these assumptions the equilibrium state for $-(\dim_H \Lambda) \log |F'|$ is unique, this follows from* [BS, Theorem 1.1]. *Note that there exist examples for which the pressure of $-(\dim_H \Lambda) \log |F'|$ is zero and there is a corresponding Gibbs measure ν which is not an equilibrium state. This occurs when $\lambda(\nu) = \infty$, see for example* [S3, p. 1757].

From now on denote $b^* := \dim_H \Lambda$. Note that since $b^* \in (0,1]$ the above implies that in Proposition 2.4 the value of t^* is less than or equal to 1. There is a wide range of examples of maps satisfying these assumptions, we discuss some of them in Section 5.

2.3. **The observable.** We will be interested in studying functions that have polynomial tails with respect to the natural geometric measure μ in Λ. For a function $\tau : \Lambda \to \mathbb{R}_{\geq 0}$ and $a \in \mathcal{A}$ we denote by τ_a the restriction of τ to the cylinder $\pi[a]$. Also, the notation $\mu(\tau > n)$ stands for $\mu(\{x \in \Lambda : \tau(x) > n\})$ and $\mu(\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\})$ for $\mu(\cup[a])$ where the union is over $\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\}$. Before stating our assumptions on τ, we introduce the following notion.

Definition 2.7. *A real valued function $\ell : \mathbb{N} \to [0,\infty)$ is said to be* slowly varying *if $\lim_{n\to\infty} \frac{\ell(\lambda n)}{\ell(n)} = 1$ for all $\lambda > 0$.*

Remark 2.8. *A slowly varying function grows (or decays) slower than any polynomial. If ℓ is slowly varying and h is a monotone increasing function such that $\liminf_{n\to\infty} \frac{\log(h(n))}{n} \in (0,\infty]$, then for all $\varepsilon > 0$ and all n sufficiently large, $h(n)^{-\varepsilon} \lesssim \ell(h(n)) \lesssim h(n)^{\varepsilon}$.*

We assume that the observable τ satisfies the following: $\sum_{j\geq 1} \mathrm{var}_j(\tau \circ \pi) < \infty$, and there is a differentiable function $\omega : \mathbb{R} \to \mathbb{R}$, strictly increasing with $\lim_{x\to\infty} \omega(x) = \infty$ and $\frac{\omega(x+1)}{\omega(x)} \leq c$ for some $c > 0$ such that

(H1) for all $\varepsilon > 0$, $1 \leq \#\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\} \lesssim e^{\varepsilon\omega(n)}$,

(H2) there exists $\beta \in (0,1)$ such that for all $n \in \mathbb{N}$,

$$\mu(\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\}) = \ell(\omega(n)) \frac{\omega'(n)}{\omega(n)^{\beta+1}},$$

(H3) there exist constants $\beta_1, \beta_2 \geq \beta$ satisfying the following: for all b close to b^*, there exists $q_0 > 0$ and such that for all $q \in [0, q_0)$, $\mu_{q,b}$ the equilibrium state for $-q\tau - b\log|F'|$, and all $n \in \mathbb{N}$,

$$e^{q\omega(n)}\omega(n+1)^{-\beta_2(b^*-b)} \lesssim \frac{\mu(\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\})}{\mu_{q,b}(\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\})} \lesssim e^{q\omega(n+1)}\omega(n)^{-\beta_1(b^*-b)}. \tag{2.3}$$

Remark 2.9. *The following are consequences of the assumptions on the observable.*

(1) (H2) implies a non-summable polynomial tail for the observable τ: for all $\varepsilon > 0$, since $\omega(n)^{-\varepsilon} \lesssim \ell(\omega(n)) \lesssim \omega(n)^{\varepsilon}$,

$$\mu(\tau > n) \asymp \int_{\omega^{-1}(n)}^{\infty} \frac{\omega'(x)}{\omega(x)^{\beta+1\pm\varepsilon}} dx = -\frac{1}{\beta \pm \varepsilon} \omega(x)^{-\beta\pm\varepsilon}\Big|_{\omega^{-1}(n)}^{\infty} \asymp n^{-\beta\pm\varepsilon}. \tag{2.4}$$

As this holds for all $\varepsilon > 0$, we conclude that $\mu(\tau > n) \asymp n^{-\beta}$.

(2) When $\omega(n) \asymp n^{\kappa}$, (H2) implies $\mu(\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\}) = \ell(\omega(n))n^{-(\beta\kappa+1)}$, and when $\omega(n) \asymp e^{rn}$ for some $r > 0$, $\mu(\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\}) = \ell(\omega(n))e^{-\beta rn}$.

(3) If $\omega(n) \asymp n^k$ for some $\kappa > 0$, then as $var_1(\tau \circ \pi) < \infty$, the $\omega(n+1)$ terms in (2.3) can be replaced by $\omega(n)$.

2.4. **Simplifications of the H conditions in terms of numbers of branches.** Under somewhat stronger assumptions we show how the conditions (H1) (H2) and (H3) are related and simplified. For a large class of systems and observables this eases the verification of the assumptions. We give two lemmas which concern the relationship between our conditions and $\#\{a : \tau_a \in [\omega(n), \omega(n+1))\}$.

Definition 2.10. *We have ω-comparable K-scaling if there exists $K > 0$ uniform such that whenever $a, a' \in \mathcal{A}$ have $\inf \tau_a, \inf \tau_{a'} \in [\omega(n), \omega(n+1))$, then $\sup_{x\in[a], y\in[a']} \frac{|F'(x)|}{|F'(y)|} = K^{\pm 1}$.*

Lemma 2.11. *Suppose that there exists $K \geq 1$, such that $\#\{a : \tau_a \in [\omega(n), \omega(n+1))\} \in [1, K]$, and we have ω-comparable K-scaling with $\omega(n) = n^\kappa$ for $\kappa > 0$. Then (H2) implies (H3) (with $\beta_{1,2} = \frac{\beta+1/\kappa}{b^*}$).*

Proof. In this case, for any $x \in [a]$ where $\inf \tau_a \in [\omega(n), \omega(n+1))$, for C_μ the Gibbs constant for μ,

$$C_\mu^{-1}|F'(x)|^{-b^*} \leq \mu\left(\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\}\right) \leq C_\mu K^2 |F'(x)|^{-b^*},$$

by (H2) and Remark 2.9(2) we have for all $\varepsilon > 0$ small, $|F'(x)|^{-b^*} \asymp \frac{1}{n^{\kappa\beta+1\pm\varepsilon}}$ and

$$\mu_{q,b}\left(\{[a] : \inf \tau_a \in [\omega(n), \omega(n+1))\}\right) \lesssim e^{-q\omega(n)}|F'(x)|^{-b} \asymp e^{-q\omega(n)} n^{-(\kappa\beta+1\pm\varepsilon)b/b^*}.$$

Therefore taking ε to 0 and the ratio of measures,

$$\frac{\mu\left(\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\}\right)}{\mu_{q,b}\left(\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\}\right)} \gtrsim e^{q\omega(n)} n^{-(\kappa\beta+1)(1-b/b^*)} = e^{q\omega(n)}\omega(n)^{-\beta'(b^*-b)},$$

where $\beta' = \frac{\beta+1/\kappa}{b^*}$. Similarly, since

$$\mu_{q,b}\left(\{[a] : \inf \tau_a \in [\omega(n), \omega(n+1))\}\right) \gtrsim e^{-q\omega(n)} n^{-(\kappa\beta+1\pm\varepsilon)b/b^*},$$

one gets

$$\frac{\mu\left(\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\}\right)}{\mu_{q,b}\left(\{a : \inf \tau_a \in [\omega(n), \omega(n+1))\}\right)} \lesssim e^{q\omega(n)}\omega(n)^{-\beta'(b^*-b)},$$

hence (H3) holds with $\beta_{1,2} = \beta'$. □

Lemma 2.12. *Suppose that both (H2) and (H3) hold and we have ω-comparable K-scaling. Then if $\omega(n)$ is polynomial (of order κ), let $c_{1,2} = \beta_{1,2}b^* - \beta$*

$$\omega'(n)\omega(n)^{c_1-1} \lesssim \#\{a : \tau_a \in [\omega(n), \omega(n+1))\} \lesssim \omega'(n)\omega(n)^{c_2-1}.$$

Proof. Let $c_n := \#\{a : \tau_a \in [\omega(n), \omega(n+1))\}$ and let $a_n := |F'(x)|^{-b^*}$ for $x \in [a]$ for some $a \in \mathcal{A}$ with $\inf \tau_a \in [\omega(n), \omega(n+1))$. Then our (H2) implies that

$$\frac{\omega'(n)}{\omega(n)^{\beta+1\pm\varepsilon}} \asymp \mu\left(\{[a] : \inf \tau_a \in [\omega(n), \omega(n+1)\}\right) \asymp c_n a_n \tag{2.5}$$

Moreover, if $\omega(n)$ is polynomial,

$$e^{-q\omega(n+1)} c_n a_n^{b/b^*} \lesssim \mu_{q,b}\left(\{[a] : \inf \tau_a \in [\omega(n), \omega(n+1))\}\right) \lesssim e^{-q\omega(n)} c_n a_n^{b/b^*}. \tag{2.6}$$

First by (H3),

$$\frac{c_n a_n}{e^{-q\omega(n)} c_n a_n^{b/b^*}} \lesssim e^{q\omega(n+1)}\omega(n)^{-\beta_1(b^*-b)}, \ \textit{i.e.}, \ a_n \lesssim \omega(n)^{-\beta_1 b^*},$$

putting this into (2.5) and taking $\varepsilon \to 0$, we get

$$c_n \gtrsim \omega'(n)\omega(n)^{\beta_1 b^* - \beta - 1}.$$

Similarly by (H3) there is $a_n \gtrsim \omega(n)^{-\beta_2 b^*}$, and put this into (2.5) and taking $\varepsilon \to 0$ we get the upper bound $c_n \lesssim \omega'(n)\omega(n)^{\beta_2 b^* - \beta - 1}$. □

Note that above, the fact that $\omega(n)$ is polynomial was only used to ensure that (2.6) holds. If it were exponential, then the discrepancy between $e^{q\omega(n)}$ and $e^{q\omega}$ for $\omega \in (\omega(n), \omega(n+1))$ may cause issues. But we shall see, for example in Example 5.4 that this is not always the case.

3. Real analyticity of the spectrum

In this section we prove that despite the fact that the multifractal decomposition induced by the Birkhoff averages is extremely complicated (each level set is dense in the repeller), the function that encodes it, $b(\cdot)$, is as regular as possible (real analytic). In order to do so we make use of the associated thermodynamic formalism. Similar results have been obtained for other dynamical systems or different classes of observables (see, for example, [IJ2, MU2, P]).

Recall that F restricted to Λ is topologically conjugated to the full-shift on $\mathcal{A} = \mathbb{N}$; abusing notations, $\sup_{x \in [a]}$ is to be understood as taking supremum over x in $\pi[a]$. The following function is our key to prove Theorem 1.1 and the real analyticity of $b(\cdot)$:

$$p : \mathbb{R}^3 \to \mathbb{R}, \quad p(\alpha, q, b) := P(q(\alpha - \tau) - b \log |F'|). \tag{3.1}$$

Let $\mathcal{S} := \left\{(q, b) \in \mathbb{R}^2 : P(-q\tau - b \log |F'|) < \infty\right\}$ denote the set where the function p is finite.

Lemma 3.1. *We have that*

$$\mathcal{S} = (\{0\} \times [b^*, \infty)) \cup ((0, \infty) \times [0, \infty)),$$

where $b^* = dim_H(\Lambda)$ is the unique solution to $P(-t \log |F'|) = 0$ (which exists by assumption (F5)).

Proof. Suppose $q > 0$. By (H1), for $\varepsilon \in (0, q)$,

$$\sum_{a \in \mathcal{A}} \exp\left(\sup -q\tau_a(x)\right) \le \sum_{k \in \mathbb{N}} \sum_{\{a : \inf \tau_a \in [\omega(k), \omega(k+1))\}} \exp(-q\omega(k)) \lesssim \sum_{k \in \mathbb{N}} \exp\left(-(q - \varepsilon)\omega(k)\right) < \infty,$$

and by Lemma 2.1, $P(-q\tau) < \infty$. Since F is uniformly expanding, for all $b \ge 0$ we have that $P(-q\tau - b \log |F'|) \le P(-q\tau) < \infty$.

Now assume that $q < 0$, then for any $b \in \mathbb{R}$, the Variational Principle implies

$$P(-q\tau - b \log |F'|) \ge h(\mu) + \int (-q\tau - b \log |F'|)\, d\mu \ge -q \int \tau \, d\mu - b\lambda(\mu).$$

By assumption $\int \tau \, d\mu = \infty$, therefore $(-\infty, 0) \times \mathbb{R} \notin \mathcal{S}$.

Now for $q = 0$, we assumed in Section 2.2 that $P(-(\dim_H \Lambda) \log |F'|) = 0$. By [I, Theorem 3.1],

$$P(-b \log |F'|) = \begin{cases} \text{non positive} & b > b^*; \\ \infty & b < b^*. \end{cases} \tag{3.2}$$

The result then follows. □

Remark 3.2. *Since $\sum_{j \ge 1} var_j(\tau \circ \pi)$, $\sum_{j \ge 1} var_j(F \circ \pi) < \infty$ and (Σ, σ) is a full-shift by* [S3, Theorem 1] *for each $(q, b) \in \mathcal{S}$ there exists a unique equilibrium state for the potential $-q\tau - b \log |F'|$. Moreover, it is a Gibbs measure.*

Let $\underline{\alpha} = \inf \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \tau(F^j x) : x \in \Lambda \right\} \in (0, \infty]$. The dimension of the level sets is obtained by the following variational formula.

Proposition 3.3. [IJ2, Theorem 3.1] *For each $\alpha > \underline{\alpha}$,*

$$dim_H(J(\alpha)) = \sup \left\{ \frac{h(\nu)}{\lambda(\nu)} : \nu \in \mathcal{M}_F, \lambda(\nu) < \infty, \int \tau \, d\nu = \alpha \right\}.$$

The following three lemmas and proposition set up the proof of Theorem 1.1 in the next section.

Lemma 3.4. *If $\alpha > \underline{\alpha}$ and $q > 0$ then $p(\alpha, q, b(\alpha)) \ge 0$.*

Proof. The proof is similar to that of [IJ2, Lemma 4.3]. It is a direct consequence of Proposition 3.3 that there exists a sequence of $F-$invariant probability measures $(\mu_n)_n$ such that for all $n \in \mathbb{N}$, $\int \tau d\mu_n = \alpha$, $h(\mu_n), \lambda(\mu_n) < \infty$, and $\lim_{n\to\infty} h(\mu_n)/\lambda(\mu_n) = b(\alpha)$. Let $0 < s_1 < s_2 < b(\alpha)$ and $q > 0$, then by Lemma 3.1, for $K := P(-q\tau - s_1 \log |F'|) < \infty$, and by the Variational Principle,

$$h(\mu_n) - q\alpha - s_1\lambda(\mu_n) = h(\mu_n) + \int (-q\tau - s_1 \log |F'|)\, d\mu_n \leq K.$$

For all n large enough we have $s_2 \leq h(\mu_n)/\lambda(\mu_n) \leq b(\alpha)$. Hence, combining these we get for all n large $\lambda(\mu_n) \leq (K + q\alpha)/(s_2 - s_1) < \infty$, so the sequence $\{\lambda(\mu_n)\}_n$ is bounded.

By the Variational Principle, since $q > 0$, we have that for all $n \in \mathbb{N}$,

$$p(\alpha, q, b(\alpha)) \geq h(\mu_n) + \int (q(\alpha - \tau) - b(\alpha) \log |F'|)\, d\mu_n.$$

Hence, by our choice of $\{\mu_n\}_n$ and boundedness of $\{\lambda(\mu_n)\}_n$,

$$p(\alpha, q, b(\alpha)) \geq \lim_{n\to\infty} \left(h(\mu_n) + q\left(\alpha - \int \tau\, d\mu_n\right) - b(\alpha)\lambda(\mu_n)\right) \geq \lim_{n\to\infty} \left(\lambda(\mu_n)\left(\frac{h(\mu_n)}{\lambda(\mu_n)} - b(\alpha)\right)\right) = 0.$$

□

Lemma 3.5. *For each $\alpha > \underline{\alpha}$ we have, that* $\inf \{p(\alpha, q, b(\alpha)) : q > 0\} = 0$ *and* $\lim_{q\to\infty} p(\alpha, q, b(\alpha)) = \infty$.

Proof. Suppose $\inf \{p(\alpha, q, b(\alpha)) : q > 0\} = C > 0$, then, using the idea of Lemma 2.2, [IJT, Lemma 3.2] implies that there exists a compact invariant set $M \subset \Lambda$ such that (i) $p_M(\alpha, q, b(\alpha)) > 0$ for all $q \in \mathbb{R}$ and (ii) $\lim_{|q|\to\infty} p_M(\alpha, q, b(\alpha)) = \infty$, where $p_M(\alpha, q, b) := P_M(q(\alpha - \tau) - b \log |F'|)$.

By analyticity and Ruelle's formula (see, for example, [MU2, Proposition 2.6.13]), (ii) implies there exists q_M such that

$$0 = \frac{\partial}{\partial q}\bigg|_{q=q_M} p_M(\alpha, q, b(\alpha)) = \int (\alpha - \tau)\, d\mu_{q_M}$$

where μ_{q_M} is the equilibrium state corresponding to $q_M(\alpha - \tau) - b(\alpha) \log |F'|$ for the restriction of F to M. Therefore $\int \tau\, d\mu_{q_M} = \alpha$, but

$$0 < p_M(\alpha, q_M, b(\alpha)) = h(\mu_{q_M}) - b(\alpha) \int \log |F'|\, d\mu_{q_M} = \lambda(\mu_{q_M}) \left(\frac{h(\mu_{q_M})}{\lambda(\mu_{q_M})} - b(\alpha)\right),$$

which is a contradiction to Proposition 3.3, so combining this with Lemma 3.4, $\inf\{p(\alpha, q, b(\alpha)) : q > 0\} = 0$ where p_M is the topological pressure restricted to M.

In order to prove the second statement, as in the proof of [IJT, Lemma 3.2(2)] with $\psi = 1$, there exists a compact F-invariant set $M \subset \Lambda$ and a measure $\tilde{\nu}$ supported on the orbit of a periodic point $\tilde{x} \in M$ such that $\alpha > \int \tau\, d\tilde{\nu}$, then by the Variational Principle

$$\lim_{q\to\infty} p(\alpha, q, b(\alpha)) \geq \lim_{q\to\infty} p_M(\alpha, q, b(\alpha)) \geq \lim_{q\to\infty} \left(\int q(\alpha - \tau)\, d\tilde{\nu} - b(\alpha)\lambda(\tilde{\nu})\right) = \infty.$$

□

Lemma 3.6. *For all $\alpha > \underline{\alpha}$, there exists $q(\alpha) \in (0, \infty)$ such that*

$$\frac{\partial}{\partial q}\bigg|_{q=q(\alpha)} p(\alpha, q, b(\alpha)) = 0. \tag{3.3}$$

In particular, $p(\alpha, q(\alpha), b(\alpha)) = 0$.

Proof. We first show that the $q(\alpha)$ exists and is strictly positive. By Lemmas 3.1 and 3.5, for q_0 large,

$$\frac{\partial}{\partial q}\bigg|_{q=q_0} p(\alpha, q, b(\alpha)) > 0.$$

Suppose there is no $q(\alpha) > 0$ such that the partial derivative with respect to q is 0. Then $p(\alpha, q, b(\alpha))$ is strictly increasing on $(0, \infty)$: if there exists an open set on which the pressure is decreasing then the derivative is negative, by analyticity and strict convexity of the pressure when it is finite, there exists a zero for derivative. Therefore by Lemma 3.5, $\lim_{q\to 0^+} p(\alpha, q, b(\alpha)) = \inf \{p(\alpha, q, b(\alpha) : q > 0\} = 0$.

Again by Lemma 3.1 $p(\alpha, 1/n, b(\alpha)) < \infty$ which implies for each n there is a unique equilibrium (Gibbs) state ν_n with respect to the potential $\frac{1}{n}(\alpha - \tau) - b(\alpha)\log|F'|$, and as $var_j(\tau/n) \leq var_j(\tau)$ for each $j \geq 1$, these Gibbs measures share the same Gibbs constant.

Also by assumption (H2) and Remark 2.9, finite 1 variations of $F' \circ \pi$ and the Gibbs property of μ, for $k \in \mathbb{N}$ large enough

$$k^{-\beta} \asymp \mu(\tau > k) \lesssim \sum_{n\in\mathbb{N}: p(n)>k} \sum_{\{a:\, \inf \tau_a \in [p(n), p(n+1))\}} \sup_{x\in[a]} |F'(x)|^{-b^*}. \tag{3.4}$$

Hence for all $n \in \mathbb{N}$, by Ruelle's formula, for all k large enough,

$$\begin{aligned}
&\frac{\partial}{\partial q}\Big|_{q=1/n} p(\alpha, q, b(\alpha)) = \int (\alpha - \tau)\, d\nu_n = \alpha + \sum_{a\in\mathcal{A}} \int_{[a]} -\tau\, d\nu_n \leq \alpha + \sum_{a\in\mathcal{A}} -\inf \tau_a \nu_n([a]) \\
&\lesssim \sum_{a\in\mathcal{A}} -\inf \tau_a \sup_{x\in[a]} \exp\left(\frac{\alpha - \tau(x)}{n} - b(\alpha)\log|F'(x)|\right) \leq \sum_{a\in\mathcal{A}} -\inf \tau_a \sup_{x\in[a]} \exp\left(\frac{\alpha - \tau(x)}{n}\right)|F'(x)|^{-b^*} \\
&\lesssim -k \exp\left(\frac{(\alpha - k)}{n} - p\left(\alpha, \frac{1}{n}, b(\alpha)\right)\right) \sum_{j\in\mathbb{N}: \omega(j)\geq k} \sum_{\{a\in\mathcal{A}: \inf \tau_a \in [\omega(j), \omega(j+1))\}} \mu([a]) \\
&\lesssim -k^{1-\beta} \exp\left(\frac{(\alpha - k)}{n} - p\left(\alpha, \frac{1}{n}, b(\alpha)\right)\right),
\end{aligned}$$

where the second to the last asymptotic inequality is by (H2) and Remark 2.9. By continuity of the pressure function and $p(\alpha, 1/n, b(\alpha)) \to 0$, for each all $n, k \in \mathbb{N}$ large enough

$$\frac{\partial}{\partial q}\Big|_{q=1/n} p(\alpha, q, b(\alpha)) \lesssim -k^{1-\beta} < 0,$$

which can be arbitrary large, implying there exists a neighbourhood to the right of 0 such that the derivative is negative, again this is a contradiction. Hence there exists $q(\alpha) > 0$ where the derivative of the pressure functional with respect to q is zero.

Now we prove the pressure at $q(\alpha) > 0$ is also zero. Let μ_α denote the equilibrium Gibbs state for the potential $q(\alpha)(\alpha - \tau) - b(\alpha)\log|F'|$. We have shown already that

$$\frac{\partial}{\partial q}\Big|_{q=q(\alpha)} p(\alpha, q, b(\alpha)) = \int (\alpha - \tau)\, d\mu_\alpha = 0,$$

in other words, $\int \tau\, d\mu_\alpha = \alpha$, so by Proposition 3.3,

$$p(\alpha, q(\alpha), b(\alpha)) = \left(\frac{h(\mu_\alpha)}{\lambda(\mu_\alpha)} - b(\alpha)\right)\lambda(\mu_\alpha) \leq 0.$$

Combining this inequality with Lemma 3.4, $p(\alpha, q(\alpha), b(\alpha)) = 0$. □

Proposition 3.7. *The functions $b(\alpha)$ and $q(\alpha)$ are real-analytic with respect to $\alpha \in (0, \infty)$, and $b(\alpha)$ is strictly increasing with* $\lim_{\alpha\to\infty} b(\alpha) = dim_H(\Lambda) = b^*$.

Proof. For analyticity, the method provided in [IJ2, Lemma 4.5] allows us to conclude that the Jacobian matrix (with respect to q, b) of the mapping

$$G : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad G(\alpha, q, b) = \begin{pmatrix} p(\alpha, q, b) \\ \frac{\partial}{\partial q} p(\alpha, q, b) \end{pmatrix}$$

evaluated at $q = q(\alpha)$, $b = b(\alpha)$ has strictly positive determinant, for all $\alpha > \underline{\alpha}$. Hence, analyticity of $q(\alpha)$ and $b(\alpha)$ follows from the Analytic Implicit Function Theorem because the pressure function is analytic when it is finite. Moreover,

$$\begin{pmatrix} q'(\alpha) \\ b'(\alpha) \end{pmatrix} = -\begin{pmatrix} \frac{\partial p}{\partial q} & \frac{\partial p}{\partial b} \\ \frac{\partial^2 p}{\partial q^2} & \frac{\partial^2 p}{\partial b \partial q} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial p}{\partial \alpha} \\ \frac{\partial^2 p}{\partial \alpha \partial q} \end{pmatrix}$$

evaluated at $\alpha, q(\alpha), b(\alpha)$. Since $\frac{\partial p}{\partial q} = 0$ at $q(\alpha)$, $\frac{\partial^2 p}{\partial q^2}$ is strictly positive by convexity of pressure function, and $\frac{\partial p}{\partial b}\Big|_{\alpha, q(\alpha), b(\alpha)} = -\int |\log|F'|d\mu_\alpha$ where μ_α is the Gibbs measure for $q(\alpha)(\alpha - \tau) - b(\alpha)\log|F'|$, we have

$$b'(\alpha) = \left(\frac{\partial^2 p}{\partial q^2} \cdot \frac{\partial p}{\partial b}\right)^{-1} \left(-\frac{\partial^2 p}{\partial q^2} q(\alpha) + \frac{\partial p}{\partial q}\frac{\partial^2 p}{\partial \alpha \partial q}\right) = q(\alpha)/\lambda(\mu_\alpha). \tag{3.5}$$

Since F is uniformly expanding, $\lambda(\mu_\alpha) > 0$, and $q(\alpha) > 0$ by Lemma 3.6, $b(\alpha)$ is strictly increasing. □

4. Proof of Theorem 1.1

Proposition 4.1. *The following holds,*

$$\lim_{\alpha \to \infty} q(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} \alpha q(\alpha) = 0.$$

Proof. To show the first equality, let $q_\infty := \limsup_{\alpha \to \infty} q(\alpha)$ and assume by contradiction that $q_\infty \in (0, \infty)$. Then, there exists $\{\alpha_n\}_n$ strictly increasing such that $\limsup_{n \to \infty} \alpha_n = \infty$ and $\lim_{n \to \infty} q(\alpha_n) = q_\infty > 0$. By Lemma 3.6, Proposition 3.7 and the continuity of the pressure function on $\mathcal{S}$, for all $\alpha > \underline{\alpha}$, we have that

$$-q(\alpha)\alpha = P(-q(\alpha)\tau - b(\alpha)\log|F'|) \in \mathbb{R}.$$

By continuity,

$$\lim_{n \to \infty} P(-q(\alpha_n)\tau - b(\alpha_n)\log|F'|) = P(-q_\infty \tau - b^* \log|F'|)$$

which is finite. However, $\lim_{n \to \infty} -\alpha_n q(\alpha_n) = -\infty$, which is a contradiction.

Similarly, suppose $q_\infty = \infty$: as in the proof of Lemma 3.5, there exists a periodic point $\tilde{x}$ of period p such that for all n large enough, $\frac{1}{p}\sum_{j=0}^{p-1} \tau(F^j \tilde{x}) < \alpha_n$ *i.e.*, $\lim_{n \to \infty} \alpha_n - \frac{1}{p}\sum_{j=0}^{p-1} \tau(F^j \tilde{x}) = \infty$. Denote by $\tilde{\nu}$ the probability measure supported on the orbit of $\tilde{x}$. Since $b(\alpha_n)$ is bounded from above for all $n \in \mathbb{N}$, hence by the definition of $q(\alpha_n)$ and Lemma 3.6, we have that

$$0 = \lim_{n \to \infty} p(\alpha_n, q(\alpha_n), b(\alpha_n)) \geq \lim_{n \to \infty} \left(\int q(\alpha_n)(\alpha_n - \tau)\, d\tilde{\nu} - b(\alpha_n)\log|(F^p)'(\tilde{x})| \right) = \infty,$$

which is a contradiction. Therefore, $\lim_{\alpha \to \infty} q(\alpha) = 0$.

Lastly, by continuity, as we have shown that $q(\alpha) \to 0$, by definition of b^* (3.2),

$$\lim_{\alpha \to \infty} q(\alpha)\alpha = -\lim_{\alpha \to \infty} p(-q(\alpha)\tau - b(\alpha)\log|F'|) = P(-b^* \log|F'|) \leq 0.$$

But since both $q(\alpha)$ and α are strictly positive, $\lim_{\alpha \to \infty} \alpha q(\alpha) = 0$. □

Lemma 4.2. *There exists $L > 0$ such that for all $\alpha > L$,*

$$b^* - b(\alpha) \asymp \int_\alpha^\infty q(t)dt. \tag{4.1}$$

Proof. For all $\alpha > \underline{\alpha}$, as $b^* = \lim_{\alpha\to\infty} b(\alpha)$, it is easy to deduce from (3.5) that $b^* - b(\alpha) = \int_\alpha^\infty q(t)/\lambda(\mu_t)dt$. Since F is uniformly expanding the Lyapunov exponent $\lambda(\mu_\alpha)$ is uniformly bounded from below, hence it suffices now to find a uniform upper bound for $\lambda(\mu_\alpha)$.

Denote by C_α the Gibbs constant for the measure μ_α; it is well-known that C_α only depends on the variations of the potential $-q(\alpha)\tau - b(\alpha)\log|F'|$, as τ is of summable variations and $\mathrm{var}_j\,(b(\alpha)\log|F'|) \leq \mathrm{var}_j(\log|F'|)$ for all $j \in \mathbb{N}$ so as $\alpha \to \infty$ the constants C_α are uniformly bounded.

By Lemma 3.6 $p(\alpha, q(\alpha), b(\alpha)) = 0$, so the Gibbs property gives $\mu_\alpha([a]) \asymp e^{q(\alpha)(\alpha-\tau_a)-b(\alpha)\log|F'_a|}$ for all $a \in \mathcal{A}$. Also we have $b(\alpha) \to b^*$ as $\alpha \to \infty$, so there exists $L > 0$ such that for all $\alpha > L$, $|b^* - b(\alpha)| < \frac{\beta}{8\beta_2}$. Since for all $y > 1$ there is $\log y \leq y^\eta/\eta$, applying (H3), for all $0 < \eta < \min\left\{\frac{b^*}{2}, \frac{\beta}{8\beta_2}, \frac{\beta}{8}\right\}$, as $\omega(n+1) \leq c\omega(n)$ and $\ell(\omega(n)) \lesssim \omega(n)^\eta$,

$$\begin{aligned}\int \log|F'|d\mu_\alpha &\asymp e^{\alpha q(\alpha)} \sum_{a\in\mathcal{A}} \log|F'_a|\mu_{q(\alpha),b(\alpha)}([a]) \lesssim \sum_{a\in\mathcal{A}} e^{-q(\alpha)\tau_a-(b(\alpha)-\eta)\log|F'_a|} = \sum_{a\in\mathcal{A}} \mu_{q(\alpha),b(\alpha)-\eta}([a]) \\ &\lesssim \sum_{n\in\mathbb{N}} e^{-q(\alpha)\omega(n)}\omega(n+1)^{\beta_2(b^*-b(\alpha)+\eta)}\frac{\ell(\omega(n))\omega'(n)}{\omega(n)^{\beta+1}} \lesssim \sum_{n\in\mathbb{N}} (c\,\omega(n))^{\beta/4}\frac{\omega'(n)}{\omega(n)^{1+\beta-2\eta}} \\ &\lesssim \sum_{n\in\mathbb{N}} \frac{\omega(n)'}{\omega(n)^{1+\beta-\beta/2}} \lesssim \int_1^\infty \frac{\omega(x)}{\omega(x)^{1+\beta/2}} = -\frac{2}{\beta}\omega(x)^{-\beta/2}\Big|_0^\infty = \frac{2}{\beta}\omega(0)^{-\frac{\beta}{2}} < \infty.\end{aligned}$$

Hence $\int \log|F'|d\mu_\alpha$ is uniformly bounded from above. □

Proposition 4.3. *For every $\varepsilon > 0$ small enough we have that*

$$q(\alpha)^{-(1-\beta-\varepsilon)} \lesssim \alpha \lesssim q(\alpha)^{-(1-\beta+\varepsilon)}.$$

Proof. As in the proof of Lemma 3.4, Lemma 3.6 and Lemma 4.2 above, for each $\alpha > \underline{\alpha}$ there exists a unique Gibbs equilibrium state μ_α for $q(\alpha)(\alpha - \tau) - b(\alpha)\log|F'|$ and $\int \tau\, d\mu_\alpha = \alpha$. By assumption there is $c > 0$ such that $\frac{\omega(x+1)}{\omega(x)} \leq c$. For arbitrary $\varepsilon > 0$ there is $L_\varepsilon \geq L$ such that for all $\alpha > L_\varepsilon$, $b^* - b(\alpha) < \min\left\{\frac{\varepsilon}{2\beta_1}, \frac{\varepsilon}{2\beta_2}\right\}$, by the Gibbs property, the regularity of τ (especially finite 1-variation) assumption (H2) and (H3), since $\ell(\omega(n)) \lesssim \omega(n)^{\varepsilon/2}$,

$$\begin{aligned}\alpha &= \sum_{a\in\mathcal{A}} \int_{[a]} \tau d\mu_\alpha \lesssim \sum_{n\in\mathbb{N}} \omega(n+1) \sum_{\{a:\inf \tau_a\in[\omega(n),\omega(n+1))\}} \mu_\alpha([a]) \\ &\lesssim \sum_{n\in\mathbb{N}} c\omega(n)e^{\alpha q(\alpha)}(c\,\omega(n))^{\beta_2(b^*-b(\alpha))}e^{-q(\alpha)\omega(n)}\frac{\ell(\omega(n))\omega'(n)}{\omega(n)^{\beta+1}} \\ &\lesssim e^{\alpha q(\alpha)} \sum_{n\in\mathbb{N}} e^{-q(\alpha)\omega(n)}\omega'(n)\omega(n)^{-(\beta-\varepsilon)} \asymp e^{\alpha q(\alpha)} \int_0^\infty e^{-q(\alpha)\omega(x)}\omega'(x)\omega(x)^{-(\beta-\varepsilon)}dx\end{aligned}$$

Then substituting $s = q(\alpha)\omega(x)$ and recalling, by Proposition 4.1, that $\alpha q(\alpha) \to 0$ as $\alpha \to \infty$, we obtain that for all $\varepsilon > 0$ and α large enough,

$$\alpha \lesssim e^{\alpha q(\alpha)} \int_0^\infty \frac{1}{q(\alpha)} e^{-s}\left(\frac{s}{q(\alpha)}\right)^{-(\beta-\varepsilon)} \asymp q(\alpha)^{-(1-\beta+\varepsilon)}\Gamma(1-\beta+\varepsilon) \tag{4.2}$$

where $\Gamma(\cdot)$ denotes the usual Gamma function, $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx$. Since $\beta \in (0,1)$, $\Gamma(1-\beta+\varepsilon)$ is bounded away from infinity hence $\alpha \lesssim q(\alpha)^{-(1-\beta+\varepsilon)}$.

Now we prove the asymptotic lower bound. Using (H3) and the substitution $s = c\,q(\alpha)\omega(x)$,

$$\begin{aligned}\alpha &\gtrsim \sum_{n\in\mathbb{N}} \omega(n)\mu_\alpha([a]) \gtrsim \sum_{n\in\mathbb{N}} e^{-q(\alpha)\omega(n+1)}\ell(\omega(n))^{-1}\omega(n)^{1+\beta_1(b^*-b(\alpha))}\omega'(n)\omega(n)^{-(1+\beta)}\\ &\gtrsim \sum_{n\in\mathbb{N}} e^{-q(\alpha)c\omega(n)}\omega'(n)\omega(n)^{-(\beta+\varepsilon)} \asymp \int_0^\infty \frac{1}{c\,q(\alpha)}e^{-s}\left(\frac{s}{c\,q(\alpha)}\right)^{-(\beta+\varepsilon)} ds\\ &\asymp q(\alpha)^{-(1-\beta-\varepsilon)}\Gamma(1-\beta-\varepsilon).\end{aligned}$$

For all $\varepsilon < \frac{1-\beta}{2}$, $\Gamma(1-\beta-\varepsilon)$ is uniformly bounded so we get $\alpha \gtrsim q(\alpha)^{-(1-\beta-\varepsilon)}$. □

Proof of Theorem 1.1. Let $\varepsilon > 0$, by Lemma 4.2, Proposition 4.3 for all $\alpha \in (L, \infty)$ we have the following asymptotic inequality:

$$b^* - b(\alpha) \asymp \int_\alpha^\infty q(t)\ dt \lesssim \int_\alpha^\infty t^{-\frac{1}{1-\beta+\varepsilon}}\ dt \asymp \alpha^{-\frac{\beta-\varepsilon}{1-\beta+\varepsilon}}. \tag{4.3}$$

As the above asymptotic inequality holds for all $\varepsilon > 0$ small, we obtain that if $x < \frac{\beta}{1-\beta}$ then,

$$\lim_{\alpha\to\infty} (b^* - b(\alpha))\,\alpha^x = 0.$$

Similarly,

$$b^* - b(\alpha) \gtrsim \int_\alpha^\infty t^{-\frac{1}{1-\beta-\varepsilon}}\ dt = \alpha^{-\frac{\beta-\varepsilon}{1-\beta-\varepsilon}},$$

so if $x > \frac{\beta}{1-\beta}$ then $\lim_{\alpha\to\infty} (b^* - b(\alpha))\,\alpha^x = \infty$. □

5. Examples

In this section we provide examples that illustrate our results. Our primary example is the induced system of the non-uniform hyperbolic interval map defined by Pomeau and Manneville.

Example 5.1 (Manneville-Pomeau map)**.** Let $\lambda > 1$ and define the *Manneville-Pomeau map* $f_\lambda : [0,1] \to [0,1]$ by

$$f_\lambda(x) = \begin{cases} x(1+2^\lambda x^\lambda) & x \in [0, 1/2),\\ 2x-1 & x \in [1/2, 1]. \end{cases}$$

Let $\tau : (1/2, 1] \to \mathbb{N}$ be the first return time map to $[1/2, 1]$, $\tau(x) := \inf\left\{j \geq 1 : f^j(x) \in [1/2, 1]\right\}$ and let $F : (1/2, 1] \to (1/2, 1]$ be the map defined by $F = f^\tau$. The map F has an invariant probability measure that is absolutely continuous with respect to the Lebesgue measure and that we denote by μ. It is well known that the map F, together with μ, satisfies (F1)–(F5) and $b^* = 1$. Consider the observable defined by τ. Clearly $\mathrm{var}_j(\tau \circ \pi) = 0$ for all $j \geq 1$, and the relevant $\omega(\cdot)$ here is $\omega(n) = n$, and (H1) holds since $\#\{\tau = n\} = 1$. By, for example the proof of [BT, Proposition 2], we have that $\mu(\tau = n) = \ell(n)n^{-(1+\frac{1}{\lambda})}$ (note that the calculation referenced gives $\ell(n)$ of the form $1 + c\log n/n$ which is uniformly bounded), then condition (H2) also holds for $\beta = 1/\lambda$. Condition (H3) follows from Remark 2.9 and Lemma 2.11, and is easily verifiable with $\beta_{1,2} = \beta+1$. Therefore by Theorem 1.1,

$$\lim_{\alpha\to\infty} (1 - \dim_H J(\alpha))\,\alpha^x = \begin{cases} 0 & \text{if } x > \frac{1}{\lambda-1};\\ \infty & \text{if } x < \frac{1}{\lambda-1}. \end{cases}$$

Below in Examples (5.2)–(5.2) we construct classes of full-branched maps that satisfies our condition for F and τ in §2.3. For every decreasing sequence of positive numbers $(a_n)_n$ such that $\sum_{i=1}^\infty a_n = 1$, we can associate a partition $(I_n)_n$ of the interval $[0,1]$ such that the length of I_n is equal to a_n. The map $F : \bigcup I_n \to [0,1]$ such that F restricted to each I_n is linear, increasing and $F(I_n) = [0,1]$ is a map that can be coded with a full shift on a countable alphabet. Clearly this system satisfies (F1)–(F5) and has $b^* = 1$. In all examples below, take $\ell(n) = 1$.

Example 5.2. Fix $r > 0$ and $s \in (0, r)$. Define $(a_n)_n$ and $(b_n)_n$ to be the sequences defined by

$$a_n = C_s n^{-(1+s)} \quad \text{and} \quad b_n = n^r,$$

where C_s is such that $1 = C_s \sum_{n=1}^{\infty} a_n \asymp C_s \int_1^\infty x^{-(1+s)} = C_s \frac{1}{s}$. Again (F1)–(F5) hold, the Lebesgue measure is the measure of maximal dimension and $b^* = 1$.

Consider the observable defined by $\tau(x) = b_n$ if $x \in I_n$, let $\omega(x) = x^r$, and since $\int_{n^{1/r}}^\infty x^{-1-s} dx \asymp n^{-s/r}$, we verify (H2) with $\beta = s/r$ and hence $\beta = s/r \in (0,1)$ and by Lemma 2.11, (H3) holds. To verify this, we notice that for $b \in (0,1)$,

$$\frac{n^{-(1+s)}}{n^{-b(1+s)}} = n^{-(1+s)(1-b)} = n^{-r\left(\beta + \frac{1}{r}\right)(1-b)},$$

so (H3) holds with $\beta_{1,2} = \beta + \frac{1}{r}$. Therefore by Theorem 1.1,

$$\lim_{\alpha\to\infty} (1 - \dim_H J(\alpha))\, \alpha^x = \begin{cases} 0 & \text{if } x > \frac{s}{r-s}; \\ \infty & \text{if } x < \frac{s}{r-s}. \end{cases}$$

Example 5.3. Let $0 < c < a$ and $b > 0$, set

$$a_n = \frac{C}{n^a},\ c_n = n^c,\ b_n = n^b \text{ such that } a, b, c > 0 \text{ and } \beta = \frac{a - c - 1}{b} \in (0,1).$$

Here C is chosen such that $\sum_n C n^{c-a} = 1$. Consider the partition of $[0,1]$ induced by $(a_n)_n$ and c_n, *i.e.*, for each n there are c_n-number of disjoint intervals of length a_n. Define F as in Example 5.2. Let $\tau : [0,1] \to \mathbb{R}$ defined by $\tau(x) = b_n$ if and only if x belongs to a partition set of length a_n. Then F satisfies all the conditions listed in §2.3 and the measure of maximal dimension is again the Lebesgue measure μ with $b^* = 1$. Let $\omega(x) = x^b$, for all $\varepsilon > 0$, $\sum_{n\in\mathbb{N}} n^c e^{-\varepsilon n^b} < \infty$ so (H1) holds, and for (H2),

$$\mu\{\tau = \omega(n)\} = \frac{C}{n^{a-c}} \asymp \frac{b n^{b-1}}{n^{(1+\beta)b}}.$$

To verify (H3), it suffices to compute that for all η close to 1,

$$\frac{n^{-(a-c)}}{n^c n^{-\eta a}} = n^{-(1-\eta)a} = n^{-b(1-\eta)(\beta + (c+1)/b)},$$

so the relevant $\beta_{1,2}$ here is $\frac{a}{b} = \beta + (c+1)/b$ and the conclusion of Theorem 1.1 holds, *i.e.*,

$$\lim_{\alpha\to\infty} (1 - \dim_H J(\alpha))\, \alpha^x = \begin{cases} 0 & \text{if } x > \frac{\beta}{1-\beta}; \\ \infty & \text{if } x < \frac{\beta}{1-\beta}. \end{cases} \tag{5.1}$$

Example 5.4. Continue the construction as in Example 5.3, let $b_n = e^n$, the number of branches with $\tau(x) = b_n$ is given by $c_n = 2^n$, and the Lebesgue measure of each partition set with $\tau = e^n$ is $a_n = \frac{K}{2^n e^{\beta n}}$ where $K = (\sum_{n\geq 1} e^{\beta n})^{-1}$. Then F satisfies all the conditions listed in §2.3 and the measure of maximal dimension is again the Lebesgue measure μ with $b^* = 1$. Clearly, the obvious choice of function here is $\omega(x) = e^x$. Then $\limsup_{x\to\infty} \frac{\omega(x+1)}{\omega(x)} = e$ and $\sum_{n\in\mathbb{N}} 2^n e^{-\varepsilon e^n} < \infty$ for all $\varepsilon > 0$, so (H1) holds. Next,

$$\mu(\{\tau \in [e^n, e^{n+1})\}) \asymp \frac{1}{e^{\beta n}} = \frac{e^n}{e^{(1+\beta)n}},$$

so (H2) is verified. Lastly for (2.3), for all b close to 1 and $q > 0$,

$$\frac{\mu(\tau = \omega(n))}{\mu_{q,b}(\tau = \omega(n))} \asymp e^{qe^n} \frac{e^{-\beta n}}{e^{-\beta b n} e^{\log 2(1-b)n}} \leq e^{qe^{n+1}} \exp\left(-n(\beta + \log 2)(1-b)\right) = e^{q\omega(n+1)} \omega(n)^{-(\beta + \log 2)(1-b)},$$

and similarly the lower bound

$$e^{q\omega(n)}\omega(n+1)^{-(\beta+\log 2)(1-b)} \lesssim e^{qe^n}\frac{e^{-\beta n}}{e^{-\beta bn}e^{\log 2(1-b)n}}.$$

So (H3) holds with $\beta_{1,2}=\beta+\log 2$, and our main theorem applies.

Example 5.5 (Lüroth expansions)**.** Every real number $x\in(0,1)$ has *Lüroth* expansion of the form:

$$x=\frac{1}{d_1}+\frac{1}{d_1(d_1-1)d_2}+\cdots+\frac{1}{d_1(d_1-1)\cdots d_{n-1}(d_{n-1}-1)d_n}+\cdots=[d_1d_2\ldots]_L$$

with $d_i\geq 2$ a positive integer for every $i\in\mathbb{N}$, see [L]. Let $F:[0,1]\to[0,1]$ be the map defined by $F(x)=n(n+1)x-n$ if $x\in[(n+1)^{-1},n^{-1})$ and $F(0)=0$. If $x=[d_1d_2\ldots]_L$ then $F(x)=[d_2d_3\ldots]_L$ (see [DK]). The Lebesgue measure, that we denote by μ, is preserved by F. Then the map F together with the Lebesgue measure satisfies (F1)–(F5) and $b^*=1$. Let $r\in\mathbb{N}$ be such that $r>1$ and $\tau:[0,1]\to\mathbb{R}$ be defined by $\tau([d_1d_2\ldots]_L)=d_1^r$. Let $\omega(x)=x^r$. Note that condition (H1) is clearly satisfied. Also

$$\mu\left(\tau\in[\omega(n),\omega(n+1))\right)=\mu([n])\asymp\frac{1}{n(n-1)}\sim\frac{1}{n^2}.$$

Thus, solving $r(\beta+1)-(r-1)=2$, (H2) holds with $\beta=\frac{1}{r}\in(0,1)$. Moreover, Lemma 2.11 implies that (H3) is satisfied. Therefore,

$$\lim_{\alpha\to\infty}\left(1-\dim_H J(\alpha)\right)\alpha^x=\begin{cases}0 & \text{if } x>\frac{\beta}{1-\beta};\\ \infty & \text{if } x<\frac{\beta}{1-\beta}.\end{cases}$$

Example 5.6 (Continued fractions)**.** Every irrational real number $x\in(0,1)$ can be written uniquely as a continued fraction of the form

$$x=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\ldots}}}=[a_1a_2a_3\ldots],$$

where $a_i\in\mathbb{N}$ (see [HW, Chapter X]). The Gauss map, $F:(0,1]\to(0,1]$, is the interval map defined by

$$F(x)=\frac{1}{x}-\left[\frac{1}{x}\right],$$

where $[x]$ denotes the integer part of the real number x. Note that for $x=[a_1a_2a_3\ldots]$ we have that $F(x)=[a_2a_3\ldots]$ (see [EW, Section 3.2]). The map F restricted to the irrational numbers together with the Gauss measure

$$\mu(A)=\frac{1}{\ln 2}\int_A\frac{1}{1+x}\,dx,$$

satisfies (F1)–(F5) (condition (F2) is satisfied for the second iterate of F, which is sufficient here) and $b^*=1$. We have that

$$\mu([n])=\frac{1}{\ln 2}\ln\left(1+\frac{1}{n(n+2)}\right)=\frac{1}{n^2}-\frac{2}{n^3}+O\left(\frac{1}{n^4}\right).$$

where $[n]=\{x=[a_1a_2\ldots]\in(0,1]:a_1=n\}=\left[\frac{1}{n},\frac{1}{n+1}\right)$. Ryll-Nardzewski [RN, Corollary 4] proved that if $p\geq 1$ then for Lebesgue almost every $x=[a_1,a_2,\ldots]\in(0,1)$,

$$\lim_{n\to\infty}\left(\frac{a_1^p+a_2^p+\cdots+a_n^p}{n}\right)^{\frac{1}{p}}=\infty.$$

The Birkhoff spectrum for these type of power mean averages was studied in [IJ2, Section 6], with less precise information on the asymptotic behaviour. Let $r>1$ and $\tau:[0,1]\to\mathbb{R}$ be defined by

$\tau([a_1 a_2 \ldots]) = a_1^r$. We have that $\int \tau d\mu = \infty$. Let $\omega(x) = x^r$. Note that condition (H1) is clearly satisfied. We also have the following estimate,

$$\mu\left(\tau \in [\omega(n), \omega(n+1))\right) = \mu([n]) \asymp \frac{1}{n^2}.$$

Thus, (H2) holds again with $\beta = \frac{1}{r}$. Moreover, Lemma 2.11 implies that (H3) is satisfied. Therefore,

$$\lim_{\alpha \to \infty} \left(1 - \dim_H J(\alpha)\right) \alpha^x = \begin{cases} 0 & \text{if } x > \frac{\beta}{1-\beta}; \\ \infty & \text{if } x < \frac{\beta}{1-\beta}. \end{cases}$$

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