ALMOST SURE CONVERGENCE OF COVER TIMES FOR ψ-MIXING SYSTEMS

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Abstract. Given a topologically transitive system on the unit interval, one can investigate the *cover time*, *i.e.*, time for an orbit to reach certain level of resolution in the repeller. We introduce a new notion of dimension, namely the *stretched Minkowski dimension*, and show that under mixing conditions, the asymptotics of typical cover times are determined by Minkowski dimensions when they are finite, or by stretched Minkowski dimensions otherwise. For application, we show that for countably full-branched affine maps, results using the usual Minkowski dimensions fail to produce a finite log limit of cover times whilst the stretched version gives an finite limit. In addition, cover times of irrational rotations are explicitly calculated as counterexamples, due to the absence of mixing.

1. Introduction

Let $\mathcal{X} \subset [0,1]$ and $f : \mathcal{X} \to [0,1]$ a topologically transitive piecewise expanding Markov map equipped with an ergodic invariant probability measure μ, we study the *cover times* for points in the repeller Λ, *i.e.*, given $x \in \Lambda$ let

$$\tau_r(x) := \inf\left\{k : \forall y \in \Lambda, \exists j \leq k : y \in d(f^j(x), y) < r\right\}.$$

The first quantitative result of expected cover times $\mathbb{E}[\tau_r]$ was obtained for Brownian motions in [M], and generalised in recent works [BJK] and [JT] for iterative function systems and one dimensional dynamical systems respectively. In [BJK], an almost sure convergence for $-\log \tau_r/\log r$ was also demonstrated for *chaos games* associated to (finite) iterated function systems, assuming the invariant measure μ supported on the attractor satisfies rapid mixing conditions. All results suggest that the asymptotic behaviour of τ_r is crucially linked to the Minkowski dimensions: let $M_\mu(r) := \min_{x\in \mathrm{supp}(\mu)} \mu(B(x,r))$, the *upper and lower Minkowski dimensions* of μ are defined respectively by

$$\overline{\dim}_M(\mu) := \limsup_{r\to 0} \frac{\log M_\mu(r)}{\log r}, \ \ \underline{\dim}_M(\mu) := \liminf_{r\to 0} \frac{\log M_\mu(r)}{\log r},$$

and simply write $\dim_M(\mu)$ when the two quantities coincide. In other words, these dimension-like quantities reflect the decay rate of the minimal μ-measure ball at scale r, and they are closely related to the box-counting dimension of the ambient space (see [FFK] for more details). In addition, the Minkowski dimensions of μ govern the asymptotic behaviour of hitting times associated to the balls which are most 'unlikely' to be visited at small scales. Our first result below gives an almost sure asymptotic growth rate of cover times in terms of the Minkowski dimensions.

Date: December 2024.

2020 *Mathematics Subject Classification.* 37A25, 37D25, 37E05, 37E10.

Key words and phrases. Cover time, exponentially ψ-mixing, irrational rotations, Minkowski dimension.

Theorem 1.1. *Let (f,μ) be a probability preserving system where f is topologically transitive, Markov and piecewise expanding. If $\overline{\dim}_M(\mu)<\infty$, then for μ-a.e. x in the repeller,*

$$\limsup_{r\to 0}\frac{\log\tau_r(x)}{-\log r}\geq\overline{\dim}_M(\mu),\quad \liminf_{r\to 0}\frac{\log\tau_r(x)}{-\log r}\geq\underline{\dim}_M(\mu).$$

If (f,μ) is exponentially ψ-mixing, then for μ-almost every $x\in\Lambda$, the inequalities above are improved to

$$\limsup_{r\to 0}\frac{\log\tau_r(x)}{-\log r}=\overline{\dim}_M(\mu),\quad \liminf_{r\to 0}\frac{\log\tau_r(x)}{-\log r}=\underline{\dim}_M(\mu).$$

In particular, it is true if the invariant measure in question is *doubling.*

Remark 1.2. *We remark that systems with finite Minkowski dimensions, or at least $\overline{\dim}_M(\mu)<\infty$, are fairly common. In particular, if μ is* doubling, i.e., *there exists constant $D>0$ such that for all $x\in supp(\mu)$ and $r>0$, $D\mu(B(x,r))\geq\mu(B(x,2r))>0$, then $\overline{\dim}_M(\mu)<\infty$.*

Proof. For each $n\in\mathbb{N}$ let $x_n\in\operatorname{supp}(\mu)$ be such that $\mu(B(x_n,2^{-n}))=M_\mu(2^{-n})$, then by the doubling property,

$$M_\mu\left(2^{-n}\right)=\mu\left(B\left(x_n,2^{-n}\right)\right)\geq D^{-1}\mu\left(B\left(x_n,2^{-n+1}\right)\right)\geq D^{-1}M_\mu\left(2^{-n+1}\right)=D^{-1}\mu\left(B(x_{n-1}2^{-n+1}\right),$$

and reiterating this one gets $M_\mu(2^{-n})\geq D^{-n+1}M_\mu(1/2)$, in other words

$$\frac{\log M_\mu(2^{-n})}{-n\log 2}\leq\frac{-(n-1)\log D+\log M_\mu(1/2)}{-n\log 2}.$$

As for all $r>0$, there is unique $n\in\mathbb{N}$ such that $2^{-n}<r\leq 2^{-n+1}$, and $\frac{\log 2^{-n}}{\log 2^{-n+1}}=1$,

$$\limsup_{r\to 0}\frac{\log M_\mu(r)}{\log r}=\limsup_{n\to\infty}\frac{\log M_\mu\left(2^{-n}\right)}{-n\log 2}\leq\frac{\log D}{\log 2}<\infty. \qquad \square$$

However, the Minkowski dimensions are not always finite due to non-doubling behaviours, or more extreme decay of $M_\mu(r)$ (see Example 3.2). Hence we need a new notion of dimension, invariant under scalar multiplication (replacing $M_\mu(r)$ by $M_\mu(cr)$ for any $c>0$ the limit does not change), to capture such decay rate in r.

Definition 1.3. Define the upper and lower *stretched Minkowski dimensions* by

$$\overline{\dim}^s_M(\mu):=\limsup_{r\to 0}\frac{\log|\log M_\mu(r)|}{-\log r},\quad \underline{\dim}^s_M(\mu):=\liminf_{r\to 0}\frac{\log\log|M_\mu(r)|}{-\log r}.$$

Those quantities should be of independent interest. Our second theorem below deals with almost sure cover times for systems in which $M_\mu(r)$ decays at stretched-exponential rates.

Theorem 1.4. *Let (f,μ) be an ergodic probability preserving system where f is topologically transitive, Markov and piecewise expanding. If $\overline{\dim}_M(\mu)=\infty$, but $0<\underline{\dim}^s_M(\mu),\overline{\dim}^s_M(\mu)<\infty$, then for μ-almost every $x\in\Lambda$,*

$$\liminf_{r\to 0}\frac{\log\log\tau_r(x)}{-\log r}\geq\underline{dim}^s_M(\mu),\quad \limsup_{r\to 0}\frac{\log\log\tau_r(x)}{-\log r}\geq\overline{dim}^s_M(\mu) \tag{1.1}$$

If (f,μ) is exponentially ψ-mixing, then for μ-almost every $x\in\Lambda$,

$$\liminf_{r\to 0}\frac{\log\log\tau_r(x)}{-\log r}=\underline{dim}^s_M(\mu),\quad \limsup_{r\to 0}\frac{\log\log\tau_r(x)}{-\log r}=\overline{dim}^s_M(\mu). \tag{1.2}$$

Layout of the paper. Basic definitions are introduced in Section 2 and we delay the proofs of the main theorems to Section 4. Several examples that satisfy Theorem 1.1 and Theorem 1.4 will be discussed in Section 3. In Section 5 we will also prove that for irrational rotations, which are known to have no mixing behaviour, Theorem 1.1 fails for almost every point when the rotations are of type η (see Definition 5.1) for some $\eta > 1$. Lastly in Section 6 we show that similar results hold for flows under some natural conditions.

2. Setup

Let $\mathcal{A}$ be a finite or countable index set, and $\mathcal{P} = \{P_a\}_{a\in\mathcal{A}}$ a collection of subintervals in $[0,1]$ with disjoint interiors covering $\mathcal{X}$. We say $f : \mathcal{X} \to [0,1]$ is a *piecewise expanding Markov map* if
(1) for any $a \in \mathcal{A}$, $f_a := f|_{P_a}$ is continuous, injective and $f(P_a)$ a union of elements in $\mathcal{P}$;
(2) there is a uniform constant $\gamma > 1$ such that for all $a \in \mathcal{A}$, $|Df_a| \geq \gamma$.

The *repeller* of f, denoted by Λ, is the collection of points with all their forward iterates contained in $\mathcal{P}$, namely

$$\Lambda := \left\{ x \in \mathcal{X} : f^k(x) \in \bigcup_{a\in\mathcal{A}} P_a \text{ for all } k \geq 0 \right\}.$$

We study the dynamics of $f : \Lambda \to \Lambda$, together with an ergodic invariant probability measure μ supported on Λ. There is a shift system associated to f: let M be an $\mathcal{A}\times\mathcal{A}$ matrix such that $M_{ab} = 1$ if $f(P_a) \cap P_b \neq \emptyset$ and 0 otherwise. f is *topologically transitive* if for all $a, b \in \mathcal{A}$, there exists k such that $M^k_{ab} > 0$. Let Σ denote the space of all *infinite admissible words*, *i.e.*,

$$\Sigma := \left\{ x = (x_0, x_1, \dots) \in \mathcal{A}^{\mathbb{N}_0} : M_{x_k, x_{k+1}} = 1, \forall\, k \geq 0 \right\}.$$

A natural choice of metric on Σ is $d_s(x, y) := 2^{-\inf\{j\geq 0:\, x_j \neq y_j\}}$, and we define the projection map $\pi : \Sigma \to \Lambda$ by

$$x = \pi\left(x_0, x_1, \dots\right) \text{ if and only if } x \in \bigcap_{i=0}^{\infty} f^{-i} P_{x_i}.$$

The dynamics on Σ is the left shift $\sigma : \Sigma \to \Sigma$ given by $\sigma(x_0, x_1, \dots,) = (x_1, x_2, \dots)$, then π defines a semi-conjugacy $f \circ \pi = \pi \circ \sigma$, and the corresponding symbolic measure $\tilde{\mu}$ of μ is given by $\mu = \pi_*\tilde{\mu}$, *i.e.*, for all Borel-measurable set $B \in \mathcal{B}([0,1])$, $\mu(B) = \tilde{\mu}\left(\pi^{-1}B\right)$.

Denote $\mathcal{P}^n := \bigvee_{j=0}^{n-1} f^{-j}\mathcal{P}$, each $P \in \mathcal{P}^n$ corresponds to an $n-$*cylinder* in Σ: let $\Sigma_n \subseteq \mathcal{A}^n$ denote all finite words of length n and for any $\mathbf{i} \in \Sigma$, the n-cylinder defined by $\mathbf{i}$ is

$$[\mathbf{i}] = [i_0, \dots, i_{n-1}] := \{y \in \Sigma : y_j = i_j,\ j = 0, \dots, n-1\},$$

then $\pi[i_0, i_1, \dots, i_{n-1}] = \bigcap_{j=0}^{n-1} f^{-j} P_{i_j} =: P_{\mathbf{i}}$.

Furthermore, (f, μ) is required to have the following mixing property.

Definition 2.1. Say μ is *exponentially ψ-mixing* if there are $C_1, \rho > 0$ and a monotone decreasing function $\psi(k) \leq C_1 e^{-\rho k}$ for all $k \in \mathbb{N}$, such that the corresponding symbolic measure $\tilde{\mu}$ satisfies: for all $n, k \in \mathbb{N}$, $\mathbf{i} \in \Sigma_n$ and $\mathbf{j} \in \Sigma^* = \bigcup_{l\geq 1} \Sigma_l$,

$$\left| \frac{\tilde{\mu}([\mathbf{i}] \cap \sigma^{-(n+k)}[\mathbf{j}])}{\tilde{\mu}[\mathbf{i}]\tilde{\mu}[\mathbf{j}]} - 1 \right| \leq \psi(k).$$

3. Examples

Theorem 1.4 is applicable to the following systems.

Example 3.1. Finitely branched Gibbs-Markov maps: let f be a topologically transitive piecewise expanding Markov map with $\mathcal{A}$ finite. f is said to be *Gibbs-Markov* if for some *potential* $\phi : \Sigma \to \mathbb{R}$ which is *locally Hölder* with respect to the symbolic metric d_s, there exists $G > 0$ and $P \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, all $x = (x_0, x_1, \dots) \in \Sigma$,

$$\frac{1}{G} \leq \frac{\tilde{\mu}([x_0, \dots, x_{n-1}])}{\exp\left(\sum_{j=0}^{n-1} \phi(\sigma^j x) - nP\right)} \leq G.$$

For maps of this kind, $|Df|$ is uniformly bounded so for each ball at scale r, it is possible to approximate any ball with finitely many cylinders of the same depth (see for example the proof of [JT, Lemma 3.2]), and by the Gibbs property of $\tilde{\mu}$, the asymptotic decay rate converges so $\dim_M(\mu)$ exists and is finite. Since Gibbs measures are exponentially ψ-mixing (see [Bow, Proposition 1.14]), by Theorem 1.1, we have

$$\lim_{r \to 0} \frac{\log \tau_r(x)}{-\log r} = \dim_M(\mu)$$

for μ-a.e. x in the repeller of f.

In the next example, when $r \to 0$ at polynomial rate, $M_\mu(r)$ decays exponentially hence $\overline{\dim}_M(\mu)$ is infinite, and the stretched Minkowski dimensions are needed.

Example 3.2. Similar to [JT, Example 7.4], consider the following class of infinitely full-branched maps: pick $\kappa > 1$ and set $c = \zeta(\kappa) = \sum_{n \in \mathbb{N}} \frac{1}{n^\kappa}$. Let $a_0 = 0$, $a_j = \sum_{j=1}^{n} \frac{1}{cj^\kappa}$ and define f by

$$\forall\, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ f(x) = cn^\kappa(x - a_{n-1}) \text{ for } x \in [a_{n-1}, a_n) =: P_n.$$

Then f is an infinitely full-branched affine map, and we can associate this map with a full-shift system on $\mathbb{N}$: $x = \pi(i_0, i_1, \dots)$ if for all $j \geq 1$, $f^j(x) \in P_{i_j}$.
Let $\omega > 1$ and construct $\tilde{\mu}$ the finite Bernoulli measure by

$$\tilde{\mu}([i_0, \dots, i_{n-1}]) = \prod_{j=0}^{n-1} \omega^{-i_j},$$

so the push-forward measure $\mu = \pi_* \tilde{\mu}$ has $\mu(P_n) = \omega^{-n}$.

Proposition 3.1. *$\overline{\dim}_M(\mu) = \infty$, but $\dim_M^s(\mu) = \frac{1}{\kappa - 1}$.*

Proof. For each $r > 0$, $M_\mu(r)$ is found near 1, then along the sequence $r_n = \frac{1}{2c} \sum_{j \geq n} j^{-\kappa} \approx \frac{1}{2c(\kappa-1)n^{\kappa-1}}$, the ball that realises $M_\mu(r_n)$ is contained in $\bigcup_{j=n}^{\infty} P_j$, hence

$$\omega^{-n} \leq M_\mu(r_n) \leq \frac{\omega^{-n}}{1 - \omega^{-1}},$$

therefore

$$\overline{\dim}_M(\mu) \geq \limsup_{n \to \infty} \frac{n \log \omega}{(\kappa - 1) \log n} = \infty,$$

whereas for all n,

$$\frac{\log n}{(\kappa - 1) \log n} \leq \frac{\log |\log M_\mu(r_n)|}{-\log r_n} \leq \frac{\log n + \log \log \omega}{(\kappa - 1) \log n - \log(2c(\kappa - 1))}.$$

As for all $r > 0$, there is unique $n \in \mathbb{N}$ such that $r_{n+1} \leq r < r_n$ while $\lim_{n \to \infty} \frac{\log r_{n+1}}{\log r_n} = 1$, one concludes with $\dim_M^s(\mu) = \frac{1}{\kappa - 1}$. $\square$

As in [JT, Example 7.4] it is very difficult for the system to cover small neighbourhoods of 1 so Theorem 1.1 says $\limsup_{r\to 0}\frac{\log\tau_r(x)}{-\log r}\geq \overline{\dim}_M(\mu)=\infty$, but since $\tilde{\mu}$ is Bernoulli hence ψ-mixing, Theorem 1.4 asserts that

$$\lim_{r\to 0}\frac{\log\log\tau_r(x)}{-\log r}=\frac{1}{\kappa-1}\quad \mu\text{-a.e.}$$

4. Proof of Theorem 1.4

The proofs in this section are adapted from those of [BJK, Proposition 3.1, 3.2]. We will only demonstrate the proofs for Theorem 1.4, *i.e.*, the asymptotics are determined by stretched Minkowski dimensions; the proofs for Theorem 1.1 are obtained by replacing all stretched exponential sequences in the proofs below by some exponential sequence, *e.g.* for a given constant $s\in\mathbb{R}$, $e^{\pm n^s}$ will be replaced by $2^{\pm ns}$.

Assuming the inequalities in (1.1), we first prove the (1.2) which requires the exponentially ψ-mixing condition.

Remark 4.1. *Assuming the conditions of Theorem 1.4, we will prove that the statements hold along the subsequence $r_n=n^{-1}$ such that for each $r>0$ there is a unique $n\in\mathbb{N}$ with $r_{n+1}<r\leq r_n$ while $\lim_{n\to\infty}\frac{\log r_{n+1}}{\log r_n}=1$ (if $\overline{\dim}_M(\mu)$ or $\underline{\dim}_M(\mu)$ are finite we choose $r_n=2^{-n}$ instead), and since $\log\tau_r(x)$ is increasing as $r\to 0$,*

$$\limsup_{n\to\infty}\frac{\log\log\tau_{r_n}(x)}{-\log r_n}=\limsup_{r\to 0}\frac{\log\log\tau_r(x)}{-\log r},$$

and similarly for liminf's.

4.1. Proof of (1.2).

Proposition 4.2. *Suppose (f,μ) is exponentially ψ-mixing, and the upper stretched Minkowski dimension of μ, $\overline{\dim}^s_M(\mu)$, is finite, then for μ-almost every $x\in\Lambda$,*

$$\limsup_{n\to\infty}\frac{\log\log\tau_r(x)}{-\log r}\leq\overline{\dim}^s_M(\mu).$$

Proof. Let $\varepsilon>0$, and for simplicity denote $\overline{\alpha}:=\overline{\dim}^s_M(\mu)$.

For any finite k-word $\mathbf{i}=x_0,\dots,x_{k-1}\in\Sigma_k$, let $\mathbf{i}^-=x_0,\dots,x_{k-2}$, *i.e.*, $\mathbf{i}$ dropping the last digit. Recall that for each $\mathbf{i}\in\Sigma^*$, $P_{\mathbf{i}}=\pi[\mathbf{i}]$, and we define

$$\mathcal{W}_r:=\{\mathbf{i}\in\Sigma^*:\ \mathrm{diam}(P_{\mathbf{i}})\leq r<\mathrm{diam}(P_{\mathbf{i}^-})\}.$$

By expansion, for each $n\in\mathbb{N}$, the lengths of the words in $\mathcal{W}_{n^{-1}}$ are bounded from above, hence we can define

$$L(n):=\frac{\log n}{\log\gamma}+1\geq\max\{|\mathbf{i}|:\mathbf{i}\in\mathcal{W}_{n^{-1}}\}.$$

Given $y\in[0,1]$ and $r>0$ such that $B(y,r)\subset\mathrm{supp}(\mu)$, define the corresponding symbolic balls by

$$\tilde{B}(y,r):=\{[\mathbf{i}]:\mathbf{i}\in\mathcal{W}_r,P_{\mathbf{i}}\cap B(y,r)\neq\emptyset\}.$$

Note that if for some $x\in P_{\mathbf{i}}$ and $[\mathbf{i}]\in\tilde{B}(y,r)$, $d(x,y)\leq r+diam(P_{\mathbf{i}})\leq 2r$ therefore

$$B(y,r)\subset\pi\tilde{B}(y,r)\subset B(y,2r).$$

Let $\mathcal{Q}_n$ be a cover of Λ with balls of radius $r_n = 1/2n$, denote the collection of their centres by $\mathcal{Y}_n$, and $\#\mathcal{Q}_n = \#\mathcal{Y}_n \le n$. Let $\tau(\mathcal{Q}_n, x)$ be the minimum time for the orbit of x to have visited each element of $\mathcal{Q}_n$ at least once,

$$\tau(\mathcal{Q}_n, x) := \min\left\{k \in \mathbb{N} : \text{ for all } Q \in \mathcal{Q}_n, \text{ exists } 0 \le j \le k : f^j(x) \in Q\right\}.$$

Then $\tau_{1/n}(x) \le \tau(\mathcal{Q}_n, x)$ for all n and all x since for all $y \in \Lambda$, there is $Q \in \mathcal{Q}_n$ and $j \le \tau(\mathcal{Q}_n, x)$ such that $f^j(x), y \in Q$ hence $d(f^j(x), y) \le 1/n$. Let $\varepsilon > 0$ be an arbitrary number and for each $k \in \mathbb{N}$, set $L'(k) = L(k) + \frac{1}{\rho}\left(k^{\overline{\alpha}+\varepsilon} - \log C_1\right)$ where C_1, ρ were given in Definition 2.1, then

$$\begin{aligned}
&\mu\left(x : \tau_{1/n}(x) > e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)\right) \le \mu\left(x : \tau(\mathcal{Q}_n, x) > e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)\right) \\
&= \mu\left(x : \exists y \in \mathcal{Y}_n : f^j(x) \notin B(y, 1/2n), \, \forall j \le e^{n^{\overline{\alpha}+\varepsilon}} L'(2n)\right) \\
&\le \mu\left(x : \exists y \in \mathcal{Y}_n : f^{jL'(4n)}(x) \notin B(y, 1/2n), \, \forall j \le e^{n^{\overline{\alpha}+\varepsilon}}\right) \qquad (4.1) \\
&= \mu\left(\bigcup_{y \in \mathcal{Y}_n} \bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} f^{-jL'(4n)}(x) \notin B(y, 1/2n)\right) \le \sum_{y \in \mathcal{Y}_n} \mu\left(\bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} f^{-jL'(2n)}(x) \notin B(y, 1/2n)\right).
\end{aligned}$$

As $\pi\left(\tilde{B}(z, r)\right) \subseteq B(z, 2r)$ for all z and all $r > 0$, using the exponentially ψ-mixing property of $\tilde{\mu}$, *i.e.*, $\psi(k) \le C_1 e^{-\rho k}$ for all $k \in \mathbb{N}$, by our choice of $L'(4n)$,

$$\begin{aligned}
&\sum_{y \in \mathcal{Y}_n} \mu\left(x : \bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} f^{-jL'(4n)}(x) \notin B(y, 1/2n)\right) \le \sum_{y \in \mathcal{Y}_{k+1}} \tilde{\mu}\left(x : \bigcap_{j=1}^{e^{n^{\overline{\alpha}+\varepsilon}}} \sigma^{-jL'(4n)}(\pi^{-1}x) \notin \tilde{B}\left(y, 1/4n\right)\right) \\
&\le \left(1 + \psi\left(\frac{1}{\rho}\left((4n)^{\overline{\alpha}+\varepsilon} - \log C_1\right)\right)\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} \sum_{y \in \mathcal{Y}_{k+1}} \left(1 - \tilde{\mu}\left(\tilde{B}\left(y, \frac{1}{2n}\right)\right)\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} \\
&\le \left(1 + e^{-n^{\overline{\alpha}+\varepsilon}}\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} \sum_{y \in \mathcal{Y}_{k+1}} \left(1 - \mu\left(B\left(y, \frac{1}{4n}\right)\right)\right)^{e^{n^{\overline{\alpha}+\varepsilon}}}. \qquad (4.2)
\end{aligned}$$

By definition of $\overline{\alpha}$, for all n large such that $\frac{\varepsilon}{4}\log n \ge \left(\overline{\alpha} + \frac{\varepsilon}{4}\right)\log 4$, there is

$$\log\left(-\log M_\mu\left(\frac{1}{4n}\right)\right) \le \overline{\alpha} + \varepsilon/4(\log 4n) \le (\overline{\alpha} + \varepsilon/2)\log n,$$

so for all $y \in \operatorname{supp}(\mu)$ and all n large,

$$\mu\left(B\left(y, \frac{1}{4n}\right)\right) \ge e^{-n^{\overline{\alpha}+\varepsilon/2}} \ge \frac{e^{-n^{\varepsilon/2}}}{e^{n^{\overline{\alpha}+\varepsilon}}}.$$

As for all $u \in \mathbb{R}$ and large k, $(1 + \frac{u}{k})^k \approx e^u$, combining (4.1) and (4.2), for some uniform constant $C_2 > 0$,

$$\begin{aligned}
\mu\left(x : \tau_{1/n}(x) > e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)\right) &\le \left(1 + e^{-n^{\overline{\alpha}+\varepsilon}}\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} \sum_{y \in \mathcal{Y}_{k+1}} \left(1 - e^{-n^{\overline{\alpha}+\varepsilon/2}}\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} \\
&\le \left(1 + e^{-n^{\overline{\alpha}+\varepsilon}}\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} n\left(1 - \frac{e^{n^{\varepsilon/2}}}{e^{n^{\overline{\alpha}+\varepsilon}}}\right)^{e^{n^{\overline{\alpha}+\varepsilon}}} \le C_2 \exp\left(\log n - e^{n^{\varepsilon/2}}\right),
\end{aligned}$$

which is clearly summable over n. Then by Borel Cantelli, for all n large enough $\tau_{1/n}(x) \leq e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)$. Since $\log L'(4n) \approx (\overline{\alpha}+\varepsilon)\log n \ll n^{\overline{\alpha}+\varepsilon}$, we have for $\mu-$a.e. $x \in \Lambda$,

$$\limsup_{n\to\infty} \frac{\log\log \tau_{1/n}(x)}{\log n} \leq \limsup_{n\to\infty} \frac{\log\log\left(e^{n^{\overline{\alpha}+\varepsilon}} L'(4n)\right)}{\log n} \leq \overline{\alpha}+\varepsilon.$$

By Remark 4.1 this upper bound for lim sup holds for all sequences decreasing to 0 and as $\varepsilon > 0$ was arbitrary, sending it to 0 one obtains that for $\mu-$a.e. $x \in \Lambda$,

$$\limsup_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r} = \limsup_{n\to\infty} \frac{\log\log \tau_{1/n}}{\log n} \leq \overline{\alpha}.$$ □

Proposition 4.3. *Suppose (f, μ) is exponentially ψ-mixing and the lower stretched Minkowski dimension of μ, $\underline{\dim}_M^s(\mu)$, is finite, then for μ-a.e. $x \in \Lambda$,*

$$\liminf_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r} \leq \underline{\dim}_M^s(\mu).$$

Proof. Again for simplicity, denote $\underline{\alpha} := \underline{dim}_M^s(\mu)$. Let $\varepsilon > 0$ and by definition of liminf there is a subsequence $\{n_k\}_k \to \infty$ such that for all k,

$$\frac{\log(-\log M_\mu(1/n_k))}{\log n_k} \leq \underline{\alpha}+\varepsilon,$$

then repeating the proof of Proposition 4.2 by replacing n by n_k everywhere, one gets that for $\mu-$almost every x,

$$\liminf_{k\to\infty} \frac{\log\log \tau_{1/n_k}(x)}{\log n_k} \leq \underline{\alpha}+\varepsilon.$$

Again send $\varepsilon \to 0$, and use the fact that liminf over the entire sequence is no greater than the liminf along any subsequence, the proposition is proved. □

4.2. **Proof of the inequalities** (1.1).

Proposition 4.4. *For μ-almost every $x \in \Lambda$,*

$$\liminf_{n\to\infty} \frac{\log\log \tau_r(x)}{-\log r} \geq \underline{\dim}_M^s(\mu).$$

Proof. We continue to use the notation $\underline{\alpha} = \underline{\dim}_M^s(\mu)$. Let $\varepsilon > 0$ be arbitrary and by definition of $\underline{\alpha}$ for all large n there exists $y_n \in \text{supp}(\mu)$ such that $\mu(B(y_n, 1/n)) \leq e^{-n^{\overline{\alpha}-\varepsilon}}$. Let

$$T(x, y, r) := \inf\left\{j \geq 0 : f^j(x) \in B(y, r)\right\},$$

so for all $n \in \mathbb{N}$ and all x, $\tau_{1/n}(x) \geq T(x, y_n, 1/n)$. Then by invariance,

$$\mu\left(x : \tau_{1/n}(x) < e^{n^{\overline{\alpha}-\varepsilon}}/n^2\right) \leq \mu\left(x : T(x, y_n, 1/n) < e^{n^{\overline{\alpha}-\varepsilon}}/n^2\right)$$

$$= \mu\left(x : \exists\, 0 \leq j < e^{n^{\overline{\alpha}-\varepsilon}}/n^2 :\ f^j(x) \in B(y_n, 1/n)\right) \leq \bigcup_{j=0}^{e^{n^{\overline{\alpha}-\varepsilon}}/n^2} \mu\left(x : f^j(x) \in B(y_n, 1/n)\right)$$

$$\leq \sum_{j=0}^{e^{n^{\overline{\alpha}-\varepsilon}}/n^2} \mu\left(f^{-j} B\left(y_n, \frac{1}{n}\right)\right) \leq \frac{e^{n^{\overline{\alpha}-\varepsilon}}}{n^2} e^{-n^{\overline{\alpha}-\varepsilon}} = \frac{1}{n^2},$$

which is summable. By Borel-Cantelli, since $2\log n \ll n^{\underline{\alpha}-\varepsilon}$, for μ-almost every x

$$\liminf_{n\to\infty} \frac{\log\log \tau_{1/n}(x)}{\log n} \geq \underline{\alpha} - \varepsilon,$$

and since $\varepsilon > 0$ was arbitrary one can send it to 0. $\square$

Similar to Proposition 4.2 and Proposition 4.3,

Proposition 4.5. *For μ-almost every $x \in \Lambda$,*

$$\limsup_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r} \geq \overline{\dim}_M^s(\mu).$$

Proof. Let $\varepsilon > 0$, then by definition of limsup there exists a subsequence $\{n_k\}_k \to \infty$ such that for all k,

$$\frac{\log\log\left(-M_\mu(1/n_k)\right)}{\log n_k} \geq \overline{\alpha} - \varepsilon.$$

Then repeating the proof of Proposition 4.4 along $\{n_k\}_k$, one gets that for μ-almost every x:

$$\limsup_{k\to\infty} \frac{\log\log \tau_{1/n_k}(x)}{\log n_k} \geq \overline{\alpha} - \varepsilon,$$

then sending $\varepsilon \to 0$,

$$\limsup_{r\to 0} \frac{\log\log \tau_r(x)}{-\log r} \geq \limsup_{k\to\infty} \frac{\log\log \tau_{1/n_k}(x)}{\log n_k} \geq \overline{\alpha}. \qquad \square$$

5. Irrational rotations

The proof of (1.2) requires an exponentially ψ-mixing rate which is a strong mixing condition, and it is natural to ask if the same asymptotic growth in Theorem 1.4 remains the same under different mixing conditions, e.g. exponentially ϕ-mixing and α-mixing, or even polynomial ψ-mixing. Although these questions are unresolved, in this section we will show that the limsup and liminf of the asymptotic growth rate can differ if the system is not mixing at all.

Let $\theta \in (0,1)$ be an irrational number and $T(x) = x + \theta \pmod 1$, and μ the one-dimensional Lebesgue measure on $[0,1)$. Then (T,μ) is an ergodic probability preserving system with $\dim_M(\mu) = 1$.

Definition 5.1. For a given irrational number θ, the *type of T_θ* is given by the following number

$$\eta = \eta(\theta) := \sup\left\{\beta : \liminf_{n\to\infty} n^\beta \|n\theta\| = 0\right\},$$

where for every $r \in \mathbb{R}$, $\|r\| = \min_{n\in\mathbb{Z}} |r-n|$.

Remark 5.2. *(See* [K]*) For every $\theta \in (0,1)$ irrational, $\eta(\theta) \geq 1$ and $\eta(\theta) = 1$ almost everywhere, but there exists irrational number with $\eta(\theta) \in (1,\infty]$, e.g. the Liouville numbers.*

For any irrational number $\theta \in (0,1)$ there is a unique continued fraction expansion

$$\theta = [a_1, a_2, \dots] := \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

where $a_i \geq 1$ for all $i \geq 1$. Set $p_0 = 0$ and $q_0 = 1$, and for $i \geq 1$ choose $p_i, q_i \in \mathbb{N}$ coprime such that

$$\frac{p_i}{q_i} = [a_1, \dots, a_i] = \frac{1}{a_1 + \frac{1}{\dots \frac{1}{a_i}}}.$$

Definition 5.3. The a_i terms are called the *i-th partial quotient* and p_i/q_i the i-th *convergent*. In particular, (see [K])

$$\eta(\theta) = \limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n}.$$

Theorem 5.4. *For any irrational rotation T_θ with $\eta(\theta) > 1$,*

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \dim_M(\mu) = 1 < \eta = \limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \ \ \mu\text{-a.e.}$$

The proof of this theorem relies on algebraic properties of the number $\eta(\theta)$, and for simplicity write η from now on.

Lemma 5.5. [KS, Fact 1, Lemma 7]

(a) $q_{i+2} = a_{i+2}q_{i+1} + q_i$ *and* $p_{i+2} = a_{i+2}p_{i+1} + p_i$.
(b) $1/(2q_{i+1}) \le 1/\left(q_{i+1} + q_i\right) < \|q_i\theta\| < 1/q_{i+1}$ *for* $i \ge 1$.
(c) If $0 < j < q_{i+1}$, *then* $\|j\theta\| \ge \|q_i\theta\|$.
(d) for $\varepsilon > 0$, *there exists uniform* $C_\varepsilon > 0$ *such that for all* $j \in \mathbb{N}$, $j^{\eta+\varepsilon}\|j\theta\| > C_\varepsilon$.

The following propositions are largely based on [KS, Proposition 6, Proposition 10].

Proposition 5.6. *For μ-a.e. x,*

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \ge \eta. \tag{5.1}$$

Proof. First we prove the following simple claim.

Claim. *The function $\varphi(x) = \limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r}$ is constant μ a.e.*

Proof of Claim. Suppose $\tau_r(x) = k$, and for any $y \ne x$, if there exists z such that for all $0 \le j \le k$, $\left|T^j y - z\right| \ge r$, then for all $0 \le j \le k$

$$\left|T^j y - x + x - z\right| = \left|T^j x - (x + z - y)\right| \ge r,$$

contradicting $\tau_r(x) = k$, then by symmetry $\tau_r(x) = \tau_r(y)$, in particular $\tau_r(x) = \tau_r(Tx)$ so $\varphi\circ T = \varphi$, and μ is (uniquely) ergodic implies φ is constant almost everywhere. □

By [KS, Proposition 10], for almost every x, y

$$\limsup_{r\to 0} \frac{\log W_{B(y,r)}(x)}{-\log r} \ge \eta,$$

where $W_E(x) := \inf\{n \ge 1 : T^n x \in E\}$ denotes the waiting time of x before visiting E. Hence there exists a set of strictly positive measure consisting of points that satisfy

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \ge \limsup_{r\to 0} \frac{\log W_{B(y,r)}(x)}{-\log r} \ge \eta,$$

since for all $y \in [0, 1)$, $\tau_r(x) \ge W_{B(y,r)}(x)$. By the claim above this holds for almost every x. □

Proposition 5.7. *For μ-a.e. x,*

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \le \eta.$$

Proof. Let $\mathcal{Q}_n := \{[2^{-n}j, 2^{-n}(j+1)) : j = 0, \dots, 2^n - 1\}$ and $\tau(\mathcal{Q}_n, x)$ the minimum time for x to have visited each element of $\mathcal{Q}_n$, again we have $\tau_{2^{-n+1}}(x) \le \tau(\mathcal{Q}_n, x)$ for all x. By Lemma 5.5 (a) and (c), $\{\|q_i\theta\|\}_i$ is a decreasing sequence, and there for each $n \in \mathbb{N}$ exists a minimal j such that $\|q_j\theta\| < 2^{-n} \le \|q_{j-1}\theta\|$, write $j = j_n$.

By [KS, Proposition 6] for all n, there is $\mu\left(W_{[0,2^{-n})} > q_{j_n} + q_{j_n-1}\right) = 0$. Notice that for all $a, b \in [0,1)$,

$$\mu\{W_{[a,a+b)}(x) = k\} = \mu\left\{\{x : W_{[0,b)}(x) = k\} + a\right\} = \mu\left\{W_{[0,b)}(x) = k\right\}, \tag{5.2}$$

as $\mu = Leb$ is translation invariant. Then by (5.2)

$$\begin{aligned}
&\mu\{\tau(\mathcal{Q}_n, x) > q_{j_n} + q_{j_n-1}\} = \mu\{x : \forall Q \in \mathcal{Q}_n : W_Q(x) > q_{j_n} + q_{j_n-1}\} \\
&= \mu\left(x : \bigcup_{Q \in \mathcal{Q}_n} \{W_Q(x) > q_{j_n-1} + q_{j_n}\}\right) \le \sum_{Q \in \mathcal{Q}_n} \mu\left(W_Q > q_{j_n-1} + q_{j_n}\right) \\
&= \sum_{j=0}^{2^n-1} \mu\left(W_{[2^{-n}j, 2^{-n}(j+1))} > q_{j_n} + q_{j_n-1}\right) = \sum_{j=0}^{2^n-1} \mu\left(W_{[0,2^{-n})} > q_{j_n} + q_{j_n-1}\right) = 0.
\end{aligned}$$

Hence by Borel-Cantelli, for all n large enough, $\tau_{2^{-n+1}}(x) \le (q_{j_n} + q_{j_n-1})$ for μ-a.e $x \in [0,1)$.

Let $\varepsilon > 0$, and by Lemma 5.5 there exists C_ε such that

$$\log(q_{j_n} + q_{j_n-1}) \le \log(2q_{j_n}) \le \log\frac{2}{\|q_{j_n}\theta\|} \le (\eta + \varepsilon)\log q_{j_n} + \log 2 - \log C_\varepsilon.$$

Again by Lemma 5.5 and our choice of j_n, for all n large enough, up to a uniform constant

$$\log\tau_{2^{-n+1}}(x) \le \log(q_{j_n} + q_{j_n-1}) < (\eta+\varepsilon)\log q_{j_n} \le -(\eta+\varepsilon)\log\|q_{j_n-1}\theta\| \le (\eta+\varepsilon)n\log 2.$$

Hence $\limsup_{n\to\infty} \frac{\log\tau_{2^{-n}}(x)}{n\log 2} \le \eta + \varepsilon$ for μ-almost every x, and send ε to 0 the proposition is proved since for each $r < 0$ there is a unique $n \in \mathbb{N}$ for which $2^{-n} < r \le 2^{-n+1}$. □

Proposition 5.8. *For μ-almost every $x \in [0,1)$,*

$$\liminf_{r\to 0} \frac{\log\tau_r(x)}{-\log r} = 1.$$

Proof. Let $\varepsilon > 0$, and using the same arguments in the last proof, *i.e.*, cover time is greater than the hitting time of the ball of smallest measure at scale r, then along the sequence $r_n = 2^{-(n+1)}$, one gets for all $[a - r_n, a + r_n) \subset [0,1)$, there is

$$\begin{aligned}
&\sum_{n\ge 1} \mu\left(\tau_{r_n}(x) < 2^{n(1-\varepsilon)}\right) \le \sum_{n\ge 1} \mu\left(W_{[a-2^{-n-1}, a+2^{-n-1})}(x) < 2^{n(1-\varepsilon)}\right) \\
&\le \sum_{n\ge 1} \sum_{k=0}^{2^{n(1-\varepsilon)}} \mu\left(T^{-k}[a - 2^{-n-1}, a + 2^{-n-1})\right) = \sum_{n\ge 1} 2^{n(1-\varepsilon)}2^{-n} = \sum_{n\ge 1} 2^{-\varepsilon n} < \infty.
\end{aligned}$$

Since for each r there is a unique n such that $r_n < r \le r_{n-1}$ while $\lim_n \frac{\log r_n}{\log r_{n-1}} = 1$ so by Borel Cantelli,

$$\liminf_{r\to 0} \frac{\log\tau_r(x)}{-\log r} = \liminf_{n\to\infty} \frac{\log\tau_{2^{-n}}(x)}{n\log 2} \ge 1 - \varepsilon,$$

and sending ε to 0 the proposition is proved.

For the upper bound of liminf, recall that $\tau(\mathcal{Q}_n, x) \geq \tau_{2^{-n}}(x)$, we can repeat the proof of Proposition 5.7, apart from that this time we choose $\{2^{-n_i}\}_i$ according to $\{q_i\}_{i\in\mathbb{N}}$: for each i, choose $n_i \in \mathbb{N}$ to be the smallest number such that

$$\|q_{i+1}\theta\| < 2^{-n_i} \leq \|q_i\theta\|,$$

hence as in Proposition 5.7,

$$\mu\left(\tau\left(\mathcal{Q}_{n_i}, x\right) > q_{i+1} + q_i\right) \leq \sum_{Q\in\mathcal{Q}_{n_i}} \mu\left(W_Q > q_{i+1} + q_i\right) = 0.$$

Again by Lemma 5.5 (b), $q_{i+1}+q_i \leq 2q_{i+1} \leq \frac{2}{\|q_i\theta\|} < 2^{n_i+1}$ by our choice of n_i, so $\lim_{i\to\infty} \frac{\log(q_i+q_{i+1})}{n_i \log 2} \leq 1$, therefore for μ-a.e. x,

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \leq \liminf_{i\to\infty} \frac{\log \tau_{2^{-n_i}}(x)}{n_i \log 2} \leq \liminf_{i\to\infty} \frac{\log \tau\left(\mathcal{Q}_{n_i}, x\right)}{n_i \log 2} \leq 1. \qquad \square$$

6. Cover time for flows

In this section we prove results analogous to Theorem 1.1 regarding cover times for the same class of flows discussed in [RT, §4].

Let $\{f_t\}_t$ be a flow on a metric space $(\mathcal{X}, d_{\mathcal{X}})$ preserving an ergodic probability measure ν, *i.e.*, $\nu\left(f_t^{-1}A\right) = \nu(A)$ for every $t \geq 0$ and A measurable. Let Ω denote the non-wandering set and define the cover time of x at scale r by

$$\tau_r(x) := \inf\left\{T > 0 : \forall y \in \Omega, \exists t \leq T : d(f_t(x), y) < r\right\}.$$

We will assume the existence of a Poincaré section $Y \subset \mathcal{X}$, and let $R_Y(x)$ denote the first hitting time to Y, *i.e.*, $R_Y(x) := \inf\{t > 0 : f_t(x) \in Y\}$, with $\overline{R} := \int R_Y d\nu < \infty$. Define the Poincaré map by (Y, F, μ) where $F = f_{R_Y}$ and μ is the induced measure on Y given by $\mu = \frac{1}{\overline{R}}\nu|_Y$. Additionally, assume the following conditions are satisfied:

(H1) $\dim_M(\mu)$ exists and is finite for (F, μ),
(H2) (Y, F, μ) is Gibbs-Markov so Theorem 1.1 is applicable for μ-almost every $y \in Y$.
(H3) $\{f_t\}_t$ has bounded speed: there exists $K > 0$ such that for all $t > 0$, $d(f_s(x), f_{s+t}(x)) < Kt$.
(H4) $\{f_t\}_t$ is topologically mixing and there exists $T_1 > 0$ such that

$$\bigcup_{0<t\leq T_1} f_t(Y) = \mathcal{X}. \tag{6.1}$$

(H5) There exists

$$C_f := \sup\left\{\operatorname{diam}(f_t(I))/\operatorname{diam}(I) : I \text{ an interval contained in } Y, 0 < t \leq T_1\right\} \in (0, \infty)$$

Remark 6.1. *The last condition is satisfied when (H3) holds and the flow is, for example, Lipschitz,* i.e.*, there exists $L > 0$ such that for all $x, y \in \mathcal{X}$,*

$$d_{\mathcal{X}}(f_t(x), f_t(y)) \leq L^t d_{\mathcal{X}}(x, y).$$

Theorem 6.2. *Let (f_t, ν) be a probability preserving flow satisfying conditions (H1)-(H5), then for ν-almost every $x \in \Omega$,*

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \geq \underline{dim}_M(\nu) - 1. \tag{6.2}$$

Furthermore, if $\overline{\dim}_M(\nu) = \dim_M(\mu) + 1$,

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \le \overline{\dim}_M(\mu) \quad \nu\text{-a.e.} \tag{6.3}$$

Proof of (6.2). This proof is analogous to those of Proposition 4.3 and [RT, Theorem 4.1]. Let

Fix some $y \in \Omega$ and $r > 0$ and consider the random variable

$$S_{T,r}(x) := \int_0^T \mathbf{1}_{B(y,r)}(f_t(x))dt,$$

and observe that by bounded speed property, for all $T > r/K$,

$$\{x : \exists 0 \le t \le T \text{ s.t. } d(f_t(x), y)) < r\} \subset \{S_{2T,2r}(x) > r/K\},$$

since if $d(f_s(x), y) < r$ for some s, then for all $t < r/K$, $d(f_{t+s}(x), y) < 2r$. Also set

$$T(x, y, r) := \inf\{t \ge 0 : f_t(x) \in B(y, r)\},$$

and similarly for all $r > 0$ and all x, z, $\tau_r(x) \ge T(x, y, r)$.

Let $\varepsilon > 0$ be arbitrary and by definition of $\underline{\alpha}$ for all large $n \in \mathbb{N}$ there exists $y_n \in \Omega$ such that $\nu(B(y_n, 2^{-n})) \le 2^{-n(\underline{\alpha}-\varepsilon)}$. By Markov's inequality, for some $\mathcal{T}_n > 0$ to be decided later,

$$\begin{aligned}
&\nu\left(x : \tau_{2^{-n}}(x) < \mathcal{T}_n\right) \le \nu\left(x : T(x, y_n, 2^{-n}) < \mathcal{T}_n\right) = \nu\left(x : \exists\, 0 \le t < \mathcal{T}_n : \, f_t(x) \in B(y_n, 2^{-n})\right) \\
&\le \nu\left(x : S_{2\mathcal{T}_n, 2^{-n+1}}(x) > r_n/K\right) \le K2^n \int_0^{2\mathcal{T}_n} \int \mathbf{1}_{B(y_n, 2^{-n+1})}(f_t(x))d\nu(x)dt \\
&\le K2^{n+1}\mathcal{T}_n \nu(B(y_n, 2^{-n+1})) \le 4K\mathcal{T}_n 2^{-(n-1)(\underline{\alpha}-\varepsilon-1)}.
\end{aligned}$$

Choosing $\mathcal{T}_n = 2^{(n-1)(\underline{\alpha}-\varepsilon-1)}/n^2$, the last term above is summable along n hence by Borel-Cantelli, for ν-almost every x

$$\liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \ge \liminf_{n\to\infty} \frac{\log \mathcal{T}_n}{n \log 2} = \underline{\alpha} - 1 - \varepsilon,$$

and since $\varepsilon > 0$ was arbitrary one can send it to 0, and by Remark 4.1 the proposition is proved. □

Note that the proof of lower bound is independent of the existence or mixing properties of the Poincaré map (Y, F, μ). For upper bound, we first prove that the cover time of the Poincaré F in Y is comparable to the cover time of the flow.

Lemma 6.3. *Define*

$$\tau_r^F(x) := \min\{n \in \mathbb{N}_0 : \forall y \in Y, \exists 0 \le j \le n : d(y, F^j x) < r\}.$$

There exists $\lambda = \frac{1}{C_f}$ *for* C_f *defined in (H5) such that* $\tau_r(x) \le T_1 + \sum_{j=0}^{\tau_{\lambda r}^F(x)} R_Y(F^j x)$.

Proof. This is adapted from the proof of [JT, Lemma 6.4] and [RT, Theorem 2.1]. F is by assumption Gibbs-Markov so one can find $\mathcal{P}(r)$, a natural partition of Y using cylinder sets with respect to F such that for each $P \in \mathcal{P}(r)$: (a) $\mathrm{diam}(P) \le r/C_f$, and (b) for all $0 < t \le T_1$, $f_t(P)$ is connected. Suppose $\tau^F_{r/C_f}(x) = k$, then the orbit $\{x, F(x), \dots, F^k(x)\}$ must have visited every element of $\mathcal{P}$, and by (6.1) for each $y \in \Omega$ there is $P \in \mathcal{P}(r)$ and $0 < s \le T_1$ such that $y \in f_s(P)$ and hence there exists $j \le k$ such that $d\left(f_s(F^j(x)), y\right) \le C_f|P| < r$. Then set $\lambda = 1/C_f$ the lemma is proved. □

Proof of (6.3). Now assume $\overline{\dim}_M(\nu) = \dim_M(\mu) + 1$. Let $\xi > 0$ be arbitrary and define the sets

$$U_{\xi,N} := \left\{ x \in Y : |R_n(x) - n\overline{R}| \le \xi n, \forall n \ge N \right\},$$

where $R_n(x) = \sum_{j=0}^{n-1} R_Y(F^j(x))$. By ergodicity, $\lim_N \mu(U_{\xi,N}) = 1$ so for N large, $\nu(U_{\xi,N}) > 0$ hence by invariance,

$$\lim_{N\to\infty} \nu\left(\bigcup_{t=0}^{\xi N} f_{-t}(U_{\xi,N}) \right) = 1. \tag{6.4}$$

Let $\varepsilon > 0$ be arbitrary. By (6.4) one can pick N^* such that for each ν typical $x \in \mathcal{X}$ there is some $t^* \le \xi N^*$ such that $f_{t^*}(x) \in Y$. By Theorem 1.1 applied to the Poincaré map and Lemma 6.3, for all sufficiently small $r > 0$ we have the following two inequalities,

$$\frac{\log \tau^F_{\lambda r}(f_{t^*}x)}{-\log \lambda r} \le \dim_M(\mu) + \varepsilon, \quad \frac{\log\left(\tau_r(x) - T_1\right)}{-\log r} \le \frac{\log\left((\overline{R} + \xi)\tau^F_{\lambda r}(f_{t^*}x)\right)}{-\log r}.$$

Then as $\lambda, \overline{R}$ are constants and ε is arbitrary, for ν-almost every x,

$$\limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \le \dim_M(\mu) = \overline{\dim}_M(\nu) - 1. \qquad \square$$

6.1. **Example: suspension semi-flows over topological Markov shifts.** In this section, we give an example of a flow for which $\dim_M(\nu) = \dim_M(\mu)+1$ is satisfied, so Theorem 6.2 is applicable.

Let $\mathcal{A}$ be a finite alphabet and M an $\mathcal{A} \times \mathcal{A}$ matrix with $\{0, 1\}$ entries, we will consider two-sided topological Markov shift systems $(\Sigma, \sigma, \phi, \mu)$, where

$$\Sigma := \left\{ x = (\dots, x_{-1}, x_0, x_1, \dots) \in \mathcal{A}^{\mathbb{Z}} : \text{ for all } j,\ x_j \in \mathcal{A} \text{ and } M_{x_j, x_{j+1}} = 1 \right\},$$

σ the usual left shift, ϕ a Hölder potential and μ is the unique Gibbs measure with respect to ϕ. We assume that $\dim_M(\mu) \in (0, \infty)$. The natural symbolic metric on Σ is $d(x, y) = 2^{-x\wedge y}$, where

$$x \wedge y = \sup\{k \ge 0 : x_j = y_j,\ \forall |j| < k\}.$$

An n-cylinder in this setting is given by $\left[x_{-(n-1)}, \dots, x_0, \dots, x_{n-1}\right] := \{y \in \Sigma,\ y_j = x_j, \forall |j| < n\}$, and it is a well-known fact that balls in Σ are precisely the cylinder sets. The left-shift map σ is bi-Lipschitz with Lipschitz constant $L = 2$. For more detailed description of the shift space, see [Bow, §1].

Let $\varphi \in L^1(\mu)$ be a positive Lipschitz function, define the space

$$Y_\varphi := \{(x, s) \in \Sigma \times \mathbb{R}_{\ge 0} : 0 \le s \le \varphi(x)\} / \sim$$

where $(x, \varphi(x)) \sim (\sigma(x), 0)$ for all $x \in I$. The suspension flow Ψ over σ is the function acts on Y_φ by

$$\Psi_t(x, s) = (\sigma^k(x), v),$$

where $k, v \ge 0$ are determined by $s + t = v + \sum_{j=0}^{k-1} \varphi(\sigma^j(x))$. The invariant measure ν for the flow Ψ on Y_φ satisfies the following: for every $g : Y_\varphi \to \mathbb{R}$ continuous,

$$\int g d\nu = \frac{1}{\int_\Sigma \varphi d\mu} \int_\Sigma \int_0^{\varphi(x)} g(x, s) ds d\mu(x). \tag{6.5}$$

The metric on Y_φ is the Bowen-Walters distance d_Y (see for example [BW]). Define another metric d_π on Y_φ: for all $(x_i,t_i)_{i=1,2}\in Y_\varphi$,

$$d_\pi((x_1,t_1),(x_2,t_2)) := \min\begin{Bmatrix} d(x,y)+|s-t|, \\ d(\sigma x,y)+\varphi(x)-s+t, \\ d(x,\sigma y)+\varphi(y)-t+s \end{Bmatrix},$$

and the following proposition says d_π is comparable to the Bowen-Walters distance.

Proposition 6.4. [BS, Proposition 17] *There exists $c=c_\pi$ such that*

$$c^{-1}d_\pi((x_1,t_1),(x_2,t_2)) \le d_Y((x_1,t_1),(x_2,t_2)) \le c\, d_\pi((x_1,t_1),(x_2,t_2)).$$

Then the Minkowski dimension of the flow-invariant measure ν is given by

Proposition 6.5. $\dim_M(\nu)=\dim_M(\mu)+1$.

Proof. The proof is based on the proof of [RT, Theorem 4.3] for correlation dimensions.

By Proposition 6.4 for all $r>0$,

$$(B(x,r/2c)\times(s-r/2c,s+r/2c))\cap Y\subset B_Y((x,s),r)$$

where B_Y denotes the ball with respect to the metric d_Y, then for all $(x,s)\in Y_\varphi$, put $\overline{\varphi}=\int_\Sigma \varphi d\mu$, then

$$\nu(B_Y((x,s),r)) \ge \nu(B(x,r/2c)\times\left(s-\frac{r}{2c},s+\frac{r}{2c}\right),$$

$$\frac{\log\nu(B_Y((x,s),r))}{\log r} \le \frac{\log\left(\frac{r}{c\overline{\varphi}}\mu\left(B(x,\frac{r}{2c})\right)\right)}{\log r}.$$

Hence $\overline{\dim}_M(\nu)=\limsup_{r\to0}\frac{\log\min_{(x,s)\in supp(\nu)}\nu(B_Y((x,s),r)}{\log r}\le\dim_M(\mu)+1$.

For lower bound, define

$$B_1 := B(x,cr)\times(s-cr,s+cr),\ \ B_2 := B(\sigma x,cr)\times[0,cr),$$
$$B_3 := \left\{(y,t): y\in B(\sigma^{-1}x,2cr),\ \text{and } \varphi(y)-cr\le t\le\varphi(y)\right\}.$$

Then as in the proof of [RT, Theorem 4.3], $B_Y((x,s),r)\subset(B_1\cup B_2\cup B_3)\cap Y_\varphi$.

For all $r>0$ and $(x,s)\in Y_\varphi$ by (6.5), and as μ is σ,σ^{-1} invariant,

$$\nu(B_1\cap Y_\varphi)=2cr\mu(B(x,cr))/\overline{\varphi},\ \ \nu(B_2,Y_\varphi)\le cr\mu(B(x,cr))/\overline{\varphi}$$
$$\nu(B_3\cap Y_\varphi)\le cr\mu(\sigma^{-1}B(x,2cr))/\overline{\varphi}=cr\mu(B(x,2cr))/\overline{\varphi}.$$

Therefore

$$\nu(B_Y((x,s),r)\le\frac{1}{\overline{\varphi}}\left(3r\mu(B(x,cr))+cr\mu(B(x,2cr))\right),$$

which is enough to conclude that $\underline{dim}_M(\nu)\ge\dim_M(\mu)+1$. $\square$

Acknowledgement

I acknowledge the support from *Chinese Scholarship Council*. I am also thankful for various comments and help from my supervisor M. Todd, as well as other comments on Section 6 from J. Rousseau.

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