Study of Minimal Sets of Circle Homeomorphisms

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Declaration

I hereby certify that this dissertation, which is approximately 12715 words in length, has been composed by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a degree. This project was conducted by me at the University of St Andrews from June 2021 to August 2021 towards fulfilment of the requirements of the University of St Andrews for the degree of MSc Mathematics,

under the supervision of

Date 15/08/2021

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Abstract

In the study of dynamical systems, circle maps are often the simplest type of one-dimensional system to be examined. However, they still admit non-trivial dynamics and can be investigated using different mathematic approaches. In principle, the dynamics of circle homeomorphism have an enriched relationship with combinatorics, which is not the primary aim of this dissertation, but we will briefly review the relation between combinatorics and the dynamical properties of circle homeomorphisms. The major focus will be around a particular object, which are, minimal sets of circle maps. We will investigate minimal sets from two different perspectives: minimal sets of a circle homeomorphism subgroup in Section II, and the minimal set of one (single) homeomorphism in Section III, with an emphasis on C^1 diffeomorphisms. These two sections can be read fairly independently, apart from one proposition in §2.1 which will be used in Section III.

In Section I, some preparatory definitions and simple results will be provided in relation to circle homeomorphisms, including lifts, orientations, and a particular space of interval maps, $S(\mathcal{J})$ with its subset of rotations $I([0,1])$. Functions in the former set $S(\mathcal{J})$ *is* identified with orientation preserving circle homeomorphisms. Then, the combinatorically defined process of the rotation number, introduced in de Melo's book, will be presented, in order to better depict the dynamical behaviour of circle homeomorphisms without fixed points. We will also take a quick look at Denjoy's theorem and a few important consequences at the end of this section, where the concept of a 'wandering interval' is also included for later sections.

In Section II, following the study of Denjoy's results, we will investigate the behaviour of minimal sets associated of a circle homeomorphism, and more generally a subgroup of $Homeo(S^1)$. We will see an interesting conjugacy between certain types of circle homeomorphism subgroups and subgroups of $PSL(2,\mathbb{R})$, an essential set of transformations in the theory of hyperbolic geometry, therefore it is possible to label such group elements parabolic, hyperbolic and elliptic. Under some

regularity called 'convergence property', a discrete circle homeomorphism subgroup is topologically conjugate to a Fuchsian group in $PSL(2, \mathbb{R})$. It is a well-known fact that every Fuchsian group G acting on the hyperbolic half-plane $\mathbb{H}^2$ attains a limit set $L(\mathrm{G})$ on the boundary $\partial\mathbb{H}^2$ (or $S^1 = \partial\mathbb{D}^2$, if we conjugate the acting group by Cayley transformation ϕ), which is unique and minimally invariant under Γ. The structure of $L(\mathrm{G})$ has only three possibilities, which coincides with those of minimal sets of subgroups of circle homeomorphisms, therefore it makes sense to pair them up Similarly, features of their minimal sets justify calling these subgroups of circle homeomorphisms, elementary, of first type and second type. Unfortunately, this section contains various results from hyperbolic geometry and group theory, and their proofs are omitted since they do not fit the general setting of this paper and one can find them in various standard textbooks of hyperbolic geometry and group theory.

In Section III, we will first consider results discussed in a work by McDuff, and later we introduce their generalisations by A. Portela, which deal with the question of whether there exists a C^1 diffeomorphism whose minimal set is topologically Cantor. These works partially answer to a problem proposed by Herman. It can be proven that for any Cantor subset $K \subset S^1$, there exists a homeomorphism with K being its unique minimal set; but this holds not in the case of a diffeomorphism, which has more desirable properties such as uniform convergence of Fourier sums. Importantly, we know the derivative of a C^1 function changes very little on a small interval of S^1, thus given a Cantor subset, the lengths of connected components in its complement need to be regulated so that its corresponding circle homeomorphism is a C^1 diffeomorphism. Two concrete examples are provided as a demonstration of minimal Cantor failing to be C^1 minimal.

This dissertation ends with a conclusion section in which the major results introduced and discussed will be reviewed quickly, together with several questions induced by the mathematical investigations done so far.

Contents

Section 1

Introduction and preliminaries

The starting point of the theory of dynamical systems is usually identified with works done by Henri Poincaré on celestial mechanics in 1890s, who intensively studied the dynamics of circle homeomorphisms in seeking classification of solutions of ODE on torus. He introduced an important quantity associated to every circle map, the *rotation number*. We will review in this section how it is defined and its contribution to conjugate maps; more specifically speaking, if $f: S^1 \to S^1$ is an orientation-preserving map without periodic points with rotation number $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$, then f is at least semi-conjugate to a rotation map $R_{\rho(f)}$ via a continuous and order-preserving h that maps orbits of f onto orbits of $R_{\rho(f)}$. Denjoy (1930) and Schwartz later (1963) in their work provided stronger statement based on Poincaré's semi-conjugacy result, that is, certain types of smoothness regularity will guarantee that a C^1 diffeomorphism is in fact topologically conjugate to an irrational rotation on the circle, and slight changes to these regularities will immediately dismiss the conjugate relationship. Proofs of these theorems require further dynamical tools, and we shall see how they inspire the study of minimal sets later.

1.1 Conventions, notations and some useful facts

Throughout this section, we will denote the smallest open interval between a, b, by (a, b), regardless of $a < b$ or $b < a$; similarly, the smallest closed interval is $[a, b]$. The length of any interval or connected subset I of the circle will be denoted as $|I|$. The complement of a subset $A \subset S^1$ is either noted as A^c or $S^1 \backslash A$. The following four items are fundamental for the analysis of circle map dynamics.

Def. 1.1 The unit circle $[0,1] \backslash \sim\, = \mathbb{R} \backslash \mathbb{Z} = \mathbb{T}^1$ is denoted as S^1.

Given a map $f: S^1 \to S^1$, $f^k = \underbrace{f \circ f \circ \ldots \circ f}_{k\ times}$.

The forward orbit of some $x \in S^1$will be denoted as $O_f(x) = \{f^k(x) |\ k \geq 0\}$, and its ω-limit set is $\omega(x) = \{y \in S^1 |\ \exists n_k \to \infty\ such\ that\ f^{n_k} \to y\}$. Notice also that since the unit circle is a compact set, $\omega(x)$ is non-empty.

Def. 1.2 Let $f: S^1 \to S^1$ be a circle map, and $\pi: \mathbb{R} \to S^1$, $\pi(x) = x\ mod\ 1$, a *lift* of f is a function $F: \mathbb{R} \to \mathbb{R}$ such that $f \circ \pi = \pi \circ F$.

Fact 1.3

(i) If F is a lift of f, then the family of lifts of f is given by $\{F + k |\ k \in \mathbb{Z}\}$. Moreover, there is a unique choice of F such that $F(0) \in [0,1)$

(ii) If f is a homeomorphism with a lift F, then f^{-1} is lifted by F^{-1}.

(iii) If f is a circle homeomorphism, then a homeomorphism F lifts f if and only if $F(x + n) = F(x) \pm n, \forall n \in \mathbb{Z}, \forall x \in \mathbb{R}$.

(proof is omitted)

Def. 1.4 $f: S^1 \to S^1$ is said to be orientation-preserving if its lift F is monotone increasing. Orientation-reversing hence corresponds to monotone decreasing lifts. Denote the set of orientation-preserving circle homeomorphisms by $Homeo^+(S^1)$, it is also a group under composition.

The first non-trivial dynamical result in this section is a strong statement on orientation-reversing maps.

Fact 1.5 An orientation-reversing circle homeomorphism has exactly two fixed points.

Proof. Let F be the unique lift of f such that $F(0) \in [0,1]$. If $F(0) = 0$, then by (iii) of **Fact 1.3**, $F(1) = -1$ since the lift is monotone decreasing, and this means by IVT there exists $y \in (0,1)$ such that $0 > F(y) = y - 1 > -1$, therefore $f \circ \pi(y) = \pi \circ F(y) = \pi(y - 1) = y$, and $\{0, y\}$ are the two fixed points of f. Otherwise, if $F(0) > 0, F(1) < 0$, by IVT there exist $x, y \in (0,1)$ such that $0 < F(x) = x$, and $F(y) = y - 1 < 0$, hence $\{x, y\}$ are the fixed points of f. ■

Fact 1.6 If F, G are lifts of f, g respectively, then $F \circ G$ lifts $f \circ g$.

Proof. $\pi \circ (F \circ G) = (\pi \circ F) \circ G = (f \circ \pi) \circ G = f \circ (g \circ \pi) = (f \circ g) \circ \pi$ by associativity of composition. This also means F^n lifts f^n. ■

Fact 1.7 If a homeomorphism $f: S^1 \to S^1$ preserves orientation and has a periodic point of period k, then every orbit is either periodic or asymptotic to a periodic orbit of order k.

Proof. Let J be a closed interval and $f : J \to J$ be a continuous injective map.

First consider the case that $f(x) > x$. Then $f^2(x) = f(f(x)) > f(x)$ since f is monotone increasing. By induction we get $f^n(x) > f^{n-1}(x)$.

$\{f^n(x)\}_{n\geq 0}$ is an increasing sequence in J, and by continuity of f, define first $y = \sup\{f^n(x)\}$, and $y = \lim_{n\to\infty} f^n(x) = f(\lim_{n\to\infty} f^{n-1}(x)) = f(y)$.

Similarly, if $f(x) < x$, the sequence $\{f^n(x)\}_{n\geq 0}$is decreasing and hence converge to $y = \inf\{f^n(x)\}_{n\geq 0}$.

In the case of a circle map, if y is a periodic point of order k, let $J = S^1 \backslash \{y\}$, and the ω-limit set of any orbit is a periodic orbit of order k. ■

The illustration handily borrowed from [2] below gives a visualization of the cases in which a circle homeomorphism has periodic points.

The right one corresponds to a orientation preserving map with 2 periodic points and the left is an orientation reversing map.

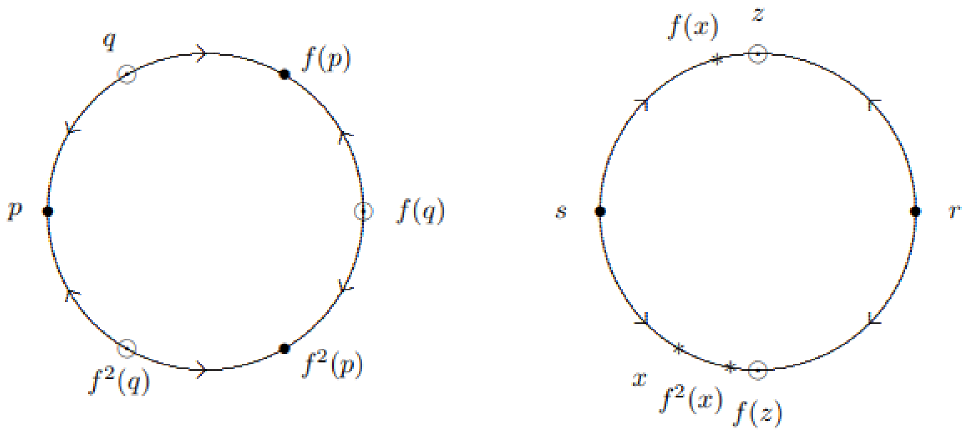


Figure 1 Circle homeomorphisms with periodic points

Moreover, by **Fact 1.3&1.6**, if f is orientation reversing, f^2 is orientation preserving, and hence every point is asymptotic to one of the two fixed points or to a periodic orbit of period 2.

Piecing these facts together one can conclude that a circle homeomorphism with periodic points implies the dynamics of any point in S^1 will approach asymptotically to the periodic points, therefore predictable in the long term. More precise statements of these maps can be derived by obtaining a better description of the set of periodic points; in fact, there is a detailed statement analogous to Sharkovskii's theorem given by X.Zhao (see [4], theorem 4.5) on circle maps with a periodic point of least period 3.

Prop. 1.8 Let $g: S^1 \to S^1$, $g(x) = x + \alpha \ (mod 1)$ for $\alpha \in (0,1)$ irrational. Then $O_g(x)$ is dense in S^1 for any $x \in S^1$.

Proof.

Claim. $\forall x \in S^1, O_g(x)$ has infinite cardinality.

Proof of claim: suppose not, then assume for some $x, \exists k < \infty \ s.t. O_g(x) = \{x, g(x), \dots, g^k(x)\}$, and hence for any $l \in \mathbb{N}$, exists $m \leq k$ such that:

$x + l\alpha \ (mod 1) = g^l(x) = g^m(x) = x + m\alpha \ (mod 1)$, which implies that $(l - m)\alpha \in \mathbb{Z}$, contradicting that α is irrational.

Now let $\varepsilon > 0$ be given and let J be a closed interval of length smaller than ε. Then there is $n \in \mathbb{N} \ s.t. \ \{g^i(J)\}_{i \geq 0}^{n}$ covers the circle: because these intervals have equal length and their interiors are disjoint.

It follows that for some $0 \leq j \leq n$, the cardinality, $|O_g(x) \cap g^j(J)| \geq 2$, i.e. $\exists l, m \in \mathbb{N}$ such that $0 < |g^l(x) - g^m(x)| < \varepsilon \Rightarrow |l - m|\alpha \ (mod 1) < \varepsilon$, so the set $\{g^{i(l-m)}([g^l(x), g^m(x)])\}_{i \in \mathbb{N}}$ covers the circle and hence every point in S^1 is ε-close to some point in $O_g(x)$. This holds for any $\varepsilon > 0$ and any $x \in S^1$. ∎

To describe the dynamics of homeomorphisms on S^1 without periodic points, Poincaré introduced a powerful tool, *rotation number*, which is invariant under topological conjugacy. This is first introduced in his study of differential equations on torus, inspired by the study of planetary orbits.

Conventionally this number is defined through a lift F of $f: S^1 \to S^1$, which is:

$$\rho(F) = \lim_{n \to \infty} \frac{F^n(x) - x}{n} \ (\text{mod} 1),$$

and this limit can be proven independent of $x \in \mathbb{R}$. However, it is also the case that this definition does not give a direct reflection of the dynamics, therefore we will consult from the combinatoric analysis of circle maps, introduced in the book by de Melo and van Strien, as another way of deriving this number.

Remark From this definition, an irrational rotation $g(x) = x + \alpha \ (mod 1)$ it can be easily proven that the rotation number of g is α.

Def. 1.9 Let $f, g: S^1 \to S^1$ be two circle homeomorphisms, call f and g are *semi-conjugate* if there exists $h: S^1 \to S^1$, a continuous surjection such that $h \circ f = g \circ h$.

If h is a homeomorphism, then f and g are said to be *topologically conjugate.* Consequently, $h^n \circ f = h \circ g^n, \forall n \in \mathbb{Z}$.

Given that a homeomorphism of S^1 preserves the structure and order of orbits, by **Prop. 1.8**, if f is topologically conjugate to an irrational rotation, then any orbit under f will necessarily be dense in the unit circle. This f often is referred to as a transitive homeomorphism. We will need one more definition before studying rotation numbers.

Def. 1.10 Let $S(\mathcal{J})$ be set of maps $f: \mathcal{J} \to \mathcal{J}$, where J is a closed interval, with the following properties:

1. f has a unique point of discontinuity $c = c(f)$.
2. $\lim_{x \to c^-} f(x)$ is the right endpoint of $\mathcal{J}$, and $\lim_{x \to c^+} f(x) = f(c)$ is the left endpoint of $\mathcal{J}$
3. f is monotone increasing in each component of $\mathcal{J} \backslash \{c\}$.
4. If the endpoints of J are $\{a, b\}$, then $f(a) = f(b)$

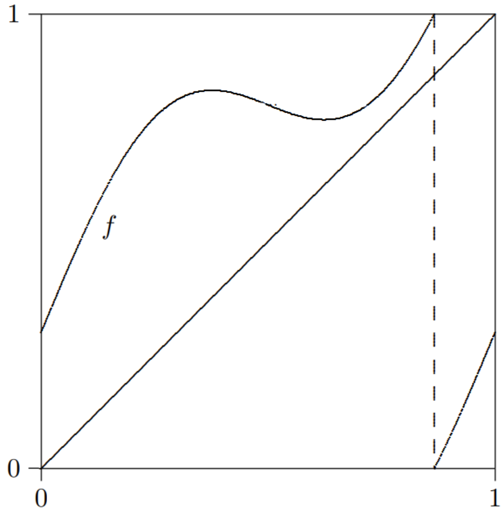


Figure 2 An example of an element in $S(\mathcal{J})$.

The set of circle homeomorphisms without fixed points (hence must be orientation-preserving) identifies with $S(\mathcal{J})$, if we identify the two endpoints of $\mathcal{J}$

(i.e. $Homeo^{+}(S^1) \cong S(\mathcal{J})$). Moreover, the set of rotations on S^1 can be identified with the subset of piecewise linear maps $I([0,1]) \subset S(\mathcal{J})$.

1.2 Rotation number and combinatorics

Now we present the construction of rotation numbers in de Melo and van Strien's book [[2], Chapter I]. There are certainly similar, precedented works done by others, yet by far this description of construction is the most detailed version.

Def. 1.11 Let $f \in S(\mathcal{J})$, let the interiors of the two connected components of J be denoted as J' *and* J''; in particular, by definition of $S(\mathcal{J})$, one of such intervals will be mapped into the other. We will let J' be such that $f(J') \subset J''$.

Define also $a(f) = \min\{i \in \mathbb{Z} |\ f^{i+1}(J') \backslash \cap J' \neq \emptyset\}$.

<u>**Remark**</u> Homeomorphisms will map intervals to intervals; moreover, they map the endpoints to endpoints (easy to prove). So, if $J = [a, b]$, and $J' = (c, b)$, then $f(J') = \big(a, f(b)\big) = (a, f(a))$, exactly by definition $S(\mathcal{J})$.

Def. 1.12 Let $f \in S(\mathcal{J})$, and suppose for some closed interval $I \subset J$, $O_f(x) \cap I$ is non-empty, then we define the *first return map* of f to interval I, $R(f): I \to I$, as $R(f)(x) = f^{k(x)}(x)$, where $k(x) = \min\{i \geq 1 |\ f^{i}(x) \in I\}$.

The following is a lemma presented in de Melo's book ([2], Lemma 1.2) in order to inductively define the sequence $a_1, a_2, \ldots$ that is closely related to both the dynamics of J' under f and the rotation number $\rho(f)$. The construction is also similar to the renormalization of f which is frequently used in other advanced works (see [2] Chapter IV).

Lemma 1.13 Let $f \in S(\mathcal{J})$ with discontinuity point $c = c(f)$, J', J'' *and* $a(f)$ the same in **Def. 2.1**, and $J(f) = cl(f^{a(f)+1} + J')$. Then

1. $a(f)$ is the smallest integer such that the closure of $J' \cup f(J') \cup \cdots \cup f^{a(f)+1}(J')$ covers the circle; these intervals lied ordered and are adjacent with disjoint interiors;

2. If $f^{a(f)}(J')$ contains c in its closure, then $f^{a(f)+1}(J') = J' = J(f), f^{a(f)+1}(c) = c$. Then $R(f)$ to $J(f)$ is $f^{a(f)+1}$, and it has fixed points in $\partial J(f)$;
3. Otherwise, $J(f)$ strictly contains J'', $R(f)$ to $J(f)$ is in $S(J(f))$; $R(f)$ maps $J'' \cap J(f)$ into J', and

$$R(f) = \begin{cases} (f|J'')^{a(f)} \circ (f|J')(x) & x \in J' \\ (f|J'')(x) & x \in J'' \cap J(f) \end{cases}$$

Proof. For (1), by assumption, $f(J') \subset J''$, and suppose J'' is $(a,c), J' = (c,b), f(J') = \big(a, f(b)\big) = (a, f(a))$ by definition of $S(\mathcal{J})$, i.e. $f(J'), J''$ share a common endpoint; similarly, $f^2(J''), f(J')$ are disjoint and share a common endpoint, and so are $f^k(J'), f^{k+1}(J')$ etc.

By definition of $a(f)$, if $f^{a(f)}(J')$ contains c in its closure, i.e. $cl(f^{a(f)}(J')) \cap (J') = \{c\}$, c is an endpoint of this interval and $f^{a(f)+1}$ maps J' onto itself and $f^{a(f)+1}(c) = c$. If c is not contained in the closure of $f^{a(f)}(J')$ then $f^{a(f)+1}(J')$ contains c in its interior, as $J(f)$ is an well-defined interval. From definition $a(f)+1$ is the smallest integer such that $f^{a(f)+1}(x) \in J(f)$ for each $x \in J' = J' \cap J(f)$. Now consider $J'' \cap J(f)$. To be more explicit, we assume that J'' is to the left of J'; if J'' is to the right of J' then simply interchange the words 'left' and 'right' in what follows. In this case, the image of $J'' \cap J(f)$ under f is equal to $J' \setminus f^{a(f)+1}(J')$. It follows that f maps $J'' \cap J(f)$ into J' and that its image contains the right endpoint of J'. Therefore, the return map $R(f)$ has such properties. ■

This lemma verifies the existence of the integer $a_1 = a(f) + 1$. In order to get $\rho(f)$, we will repeat this process unless a periodic point of f occurs.

Let $J_0 = J, \phi_0: J_0 \to J_0, \phi_0 = f$, we let $a_1 = \infty$ if f has fixed points and otherwise we define

$$a_1 = \begin{cases} a(f) + 1 & \text{if } J' \text{ is to the right of } J'' \\ 1 & \text{if } J' \text{ is to the left of } J'' \end{cases}$$

$$J_1 = \begin{cases} J(\phi_0) & \text{if } J' \text{ is to the right of } J'' \\ J & \text{if } J' \text{ is to the left of } J'' \end{cases}$$

$$\phi_1 = \begin{cases} R(f) & \text{if } J' \text{ is to the right of } J'' \\ f & \text{if } J' \text{ is to the left of } J'' \end{cases}$$

We want to make sure ϕ_1 maps the left component of $J_1 \setminus \{c\}$ into the right component.

Now suppose that $n \geq 2$ and that $J_1, \dots, J_{n-1}, \phi_1, \dots, \phi_{n-1}$ are defined, and that $\phi_{n-1}: J_{n-1} \to J_{n-1}$ has no fixed points. Then define the interval J_n, the return map ϕ_n to J_n, and the integer a_n inductively by

$$J_n = J(\phi_{n-1}),\ \phi_n = R(\phi_{n-1}): J_n \to J_n$$

$$a_n = a(\phi_{n-1})$$

On the other hand, if $\phi_{n-1}: J_{n-1} \to J_{n-1}$ has fixed points then we let $a_n = \infty$ and we stop the inductive definition.

If J' is to the right of J'' we have:

$$a_1 = a(f) + 1,\ \phi_1 = R(f)$$
$$a_n = a(\mathcal{R}^{n-1}(f)), \phi_n = R^n(f) \text{ for all } n = 2,3, \dots$$

Otherwise:

$$a_1 = 1,\ \phi_1 = f$$

$$a_n = a(R^{n-2}(f)),\ \phi_n = R^{n-1}(f) \text{ for all } n = 2,3, \dots$$

ϕ_n is the first return map of f to J_n. In particular, if f has no periodic points the construction never stops. If J' is to the left of J'' then $a_1 = 1, J_1 = J$. It follows that ϕ_1 always maps the left component of $J_1 \setminus \{c\}$ into the right component. Let J_n' be the interior of the left component of $J_n \setminus \{c\}$ if n is odd and of the right component if n is even. Denote the interior of the other component of $J_n \setminus \{c\}$ by J_n''. From the previous lemma, the role of the right and left components of $J_n \setminus \{c\}$ is interchanged in each step of the induction.

More precisely, $J_n' = J_{n-1}'' \cap J_n$ and $J_n'' = J_{n-1}' \cap J_n = J_{n-1}'$.
By construction ϕ_n maps J_n' into J_n'' for all $n \geq 1$, as $\phi_n \in S(J_{n-1})$ for which ϕ_n is defined. Also, $\phi_1 \mid J_1' = f$ and $\phi_1 \mid J_1'' = f^{a_1}$. Also, by induction one can prove:

$$\phi_n \mid J_n'' = (\phi_{n-1} \mid J_{n-1}'')^{a(\phi_{n-1})} \circ (\phi_{n-1} \mid J_{n-1}')$$

and

$$\phi_n | J_n' = \phi_{n-1} | J_{n-1}''$$

Remark Notice how the intervals $\{J_n\}_{n\geq 1}$ are 'shrinking' near the unique discontinuity point $c = c(f)$, so intuitively the rotation number below is closely related to the dynamics of c under f.

Def. 1.14 A *continued fraction*, $\alpha = [0;\ a_1, a_2, \dots, a_n]$ is defined as:

$$[0; a_1, a_2, \dots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}}$$

From the construction above, the *rotation number* of $f: S^1 \to S^1$, a homeomorphism without fixed points, $\rho(f)$, is given by:

$\rho(f) = [0; a_1, a_2, \dots, a_n, \dots.]$ if the procedure described before never stops,

or $[0; a_1, a_2, \dots, a_k], for\ k = \max\{j \in \mathbb{N} |\ a_j \neq \infty\}$ if f has a periodic point of period k in the interval J_k.

Remark It can be checked that the nth convergent, $r_n = [0; a_1, a_2, \dots, a_n] = p_n/q_n$ satisfies the Fibonacci type recursions (see [22] for more advanced analysis of this number).

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad p_0 = 0, p_1 = 1$$

$$q_n = a_n q_{n-1} + q_{n-2}, \qquad q_0 = 1, q_1 = a_1$$

This continued fraction is necessarily an irrational number if and only if the sequence $\{a_n\}_{n\geq 1}$ is infinite; otherwise $\rho(f) \in \mathbb{Q}$. The following is a classical result that Poincaré has proven in his work which relates this number with conjugacy classification (see [23] Chapitre XV).

Theorem 1 Let f be an orientation-preserving circle homeomorphism without periodic points, then there is a unique rotation $R_{\rho(f)} \in I([0,1])$ such that there is a surjective, continuous, and monotone function h, with

$$h \circ f = R_{\rho(f)} \circ h$$

Where $R_{\rho(f)}(x) = x + \rho(f) \pmod 1$, and h is a semi-conjugacy between these two maps.

Proof of this theorem will be omitted, one can find a structured proof that requires great machinery using the construction above and symbolic dynamics again in [2] Chapter I, or the more traditional ways via lifts. In any case, we will take this result as a starting point for further investigation of circle maps.

A natural question is how semi-conjugacy differs from topological conjugacy, where h is required to be a homeomorphism; they are, in fact, quite different in the sense that structure of dynamics may not be perfectly preserved by semi-conjugacy, which is exactly the reason that Denjoy has shown in [1] that irrational rotation numbers allow the existence of *wandering sets*, a concept hugely important in the study of minimal sets in Section II&III. In any case, we can prove the following general statement first.

Lemma 1.15 Let $f: X \to X$, $g: Y \to Y$ where X, Y are compact metric spaces, and suppose exists a semi-conjugacy $h: X \to Y$ such that $h \circ f = g \circ h$, then if $\tilde{X} = X/\sim$, where $x \sim y$ if and only if $h(x) = h(y)$, then f descends to the function $\tilde{f}: \tilde{X} \to \tilde{X}$, which is conjugate to g.

Proof. Let's first check if $\tilde{f}$ is well-defined. Suppose $h(x) = h(y)$, then by definition of semi-conjugacy, $h \circ f(x) = g \circ h(x) = g \circ h(y) = h \circ f(y)$, so $f(x) \sim f(y)$ as required.

Now to construct the conjugacy, let $\tilde{h}: \tilde{X} \to Y$, defined by $\tilde{h}(x) = h(q^{-1}(x))$, where $q: X \to \tilde{X}$ is the quotient map. It is also well-defined as $q(x) = q(y)$ if and only if $x \sim y \leftrightarrow h(x) = h(y)$. $\tilde{h}$ is injective, follows from the relation given on X; it is also surjective because the semi-conjugacy h is itself surjective. Therefore $\tilde{h}$ is a bijection. $\tilde{h}$ is also continuous because given open set $A \subset Y$, $\tilde{h}^{-1}(A) = q(h^{-1}(A))$ by definition, where q is an open mapping and h is a continuous mapping, so $h^{-1}(A)$ is open, therefore so is $\tilde{h}^{-1}(A)$. $\tilde{X}$ is the quotient space of a compact space hence itself compact, and a continuous bijection between such spaces will always be a homeomorphism, so $\tilde{f}$ is indeed conjugate to g. ■

Remark

1. Perhaps it will be easier to see how rational rotation number leads to periodic points using lifts: by **1.6** F^m lifts f^m, hence $\rho(F^m) = \lim_{k\to\infty} \frac{F^{mk}(x)-x}{k} = \lim_{k\to\infty} m \cdot \frac{F^{mk}(x)-x}{mk} = m \lim_{n} \frac{F^{n}(x)-x}{n} = m\rho(F)$, and it is easy to see that $\rho(F^m)\ (mod\ 1) = \rho(f^m) = 0$ implies f^m has a fixed point.

2. In **Lemma 1.15**, Y needs only to be Hausdorff.

3. The idea of collapsing S^1 into a smaller circle depending on the dynamics of the circle homeomorphism that is semi-conjugate to some irrational rotation, will be revisited in §3.1, **Prop. 3.1**.

This lemma tells us that if $f \in Homeo^+(S^1)$ has an irrational rotation number, then its dynamics will 'contain' the dynamics of an irrational rotation R, and by **Prop. 1.8** any orbit under R is dense in S^1, i.e. $cl\big(O_R(x)\big) = S^1 = \omega(x), \forall x \in S^1$, which is certainly not true for all such f with irrational rotation number; there are infinitely many counterexamples and one can find in Denjoy's original works, or two structured counterexamples in [5] Part 1 and [9]. We will now introduce Denjoy's theorems.

1.3 Denjoy's results

Def. 1.16 Call J a *wandering interval* of map f if

i. $J, f(J), f^2(J), \cdots$ are pairwise disjoint, and

ii. $\bigcup_{x\in J} \omega(x)$ is not a single periodic orbit.

This turns out to be a useful and universal tool in analyses of other dynamical systems. Immediately we know that any irrational rotation map does not have wandering intervals, and a homeomorphism $f: S^1 \to S^1$ with periodic points cannot have wandering intervals (this follows from **Fact 1.7**). In fact, f without periodic points is conjugate to an irrational rotation if and only if f does not have a wandering interval. Based on this fact, the following two theorems proven by Denjoy in [1] give classification of $\mathcal{C}^1$ diffeomorphisms.

Def. 1.17 A function $f: I \to \mathbb{R}$ on the closed interval I is said to have *bounded variation* if $\sup_{P\in\mathcal{P}} Var(f,P) < \infty$, where $P = \{x_0 < x_1 < \cdots < x_n\}$ is a partition on I, $\mathcal{P}$ is the set of all partitions on I, and $Var(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$.

Theorem 2 Suppose f is a $\mathcal{C}^1$ diffeomorphism with irrational rotation number and the derivative Df (and the derivative of f^{-1}) have bounded variation, then f has no wandering intervals and hence is topologically conjugate to the rigid rotation $R_{\rho(f)}(x) = x + \rho(f) \ mod\ 1$.

Theorem 3 There exist $\mathcal{C}^1$diffeomorphisms f with irrational rotation number such that f has non-empty wandering intervals.

Proof of the first classical theorem will be included here, whereas the second one can be proven by providing any concrete example so, as before, can be found in [5] Chapter I; M.Herman [6] even provided Denjoy counterexamples for $C^{1+\lambda}$ diffeomorphisms, where $\lambda < 1$. For **Theorem 2**, we will follow the proof given in the book by de Melo and van Strien, consistent with **Lemma 1.13** and its consequent construction; another component of the proof is the distortion of derivatives. Certainly there are more and maybe simpler ways of proving **Theorem 2**, for example there is a one by R.S.Mackay's [7] using commuting pair system, or another in M.Brin and G.Stuck's book (see [12]chapter 7) via Haar measure.

Recall from the construction of $\rho(f)$,

$$\phi_n \mid J_n' = f^{q_{n-1}} \text{ and } \phi_n \mid J_n'' = f^{q_n} \qquad (\blacktriangle)$$

where q_n is defined inductively by

$$q_0 = 1, q_1 = a_1$$
$$q_{n+1} = q_{n-1} + a_{n+1}q_n \text{ for } n \geq 1$$

Since $\phi_n: J_n \to J_n$ is a map in $S(J_n)$ by **Lemma 1.13**, $J_n = [f^{q_{n-1}}(c), f^{q_n}(c)]$, which is an interval that contains c in its interior, otherwise f has a periodic point. Therefore, $J_n' = (c, f^{q_n}(c))$, and $J_n'' = \big(c, f^{q_{n-1}}(c)\big)$.

Furthermore, we can make the statement that for $n \geq 1, 1 \leq j \leq q_{n+1}$:

$f^j(c) \in J_n$ if and only if $j = q_{n-1} + iq_n$, for some $i \in \{0, a_1, \dots, a_{n+1}\}$.

Lemma 1.16 *Suppose* $f \in S(\mathcal{J})$ has no periodic points. The union of $\bigcup_{i=0}^{q_{n-1}-1} f^i(J_n')$ and $\bigcup_{i=0}^{q_n} f^i(J_n'')$ tiles the interval $\mathcal{J}$ (or S^1), and these intervals are disjoint.

Proof. The first return map of f to $J_n' \cup J_n''$ is equal to $f^{q_{n-1}}$ on J_n' and f^{q_n} on J_n'' by ($\blacktriangle$) and the images of J_n', J_n'' under the first return map are disjoint, and their orbits before the first return time as well. Now take $x \in \mathcal{J}$ and let $k = \min\{i \in \mathbb{Z} | f^{-i}(x) \in J_n = cl(J_n' \cup J_n'')\}$. Such an integer k certainly exists because the union of $J_n', \dots, f^{q_{n+1}}(J_n')$ covers the interval in the same sense as in **Lemma 1.13**.

So either: if $f^{-k}(x)$ is in the closure of J_n' then because J_n' returns within time q_{n-1} to the closure of $J_n' \cup J_n''$, namely $J_n$one has $k < q_{n-1}$ and therefore x is in the closure of $f^k(J_n')$; or: x is in the closure of $f^k(J_n'')$ for some $k < q_n$. So in both cases, the result holds true. ■

Def. 1.17 Suppose $f: N \to N$ is a C^α map with $\alpha \geq 1$ on$N = S^1$ *or* $[0,1]$, and suppose that $I \subset X$ is an interval such that $Df(x) = f'(x) \neq 0, \forall x \in I$, then the *distortion* of f in I is defined as:

$$Dist(f, I) = \sup_{x,y \in I} \log \frac{|Df(x)|}{|Df(y)|}$$

Lemma 1.18 Let $f: N \to N$ be a C^1 map such that $f'(x) \neq 0, \forall x \in N$, and the map $x \mapsto \log\ |Df(x)|$ has bounded variation $\leq C, C \in \mathbb{R}$. Then for any interval $T \subset N$ such that $N, f(N), \ldots, f^{n-1}(N)$ are disjoint, we have

$$Dist(f^n, T) \leq C\ diam(N)$$

Proof. By chain rule,

$$log \frac{|Df^n(x)|}{|Df^n(y)|} = \sum_{i=0}^{n-1} log \frac{|Df(f^i(x))|}{|Df(f^i(y))|}$$

Given that $f^i(x), f^i(y) \in f^i(T)\ for\ all\ x, y \in T$, taking supremum at both sides we will get $Dist(f^n, T) \leq \sum_{i=0}^{n-1} Dist(f, f^i(T))$.

Also, $\forall x, y \in f^i(T), \log \frac{|Df^i(x)|}{|Df^i(y)|} = \log|Df(x)| - \log|Df(y)| \leq Var\left(f|_{f^i(T)}\right) \leq C \cdot \left|f^i(T)\right|$, and given disjointness of T, $\sum_{i=0}^{n-1} \left|f^i(T)\right| \leq diam(N)$, combining with result about will prove the result. ■

We will need one more dynamical result before marching into the proof of Denjoy's theorem, that is, the limiting behaviour of wandering intervals of circle homeomorphisms.

Lemma 1.19 (Contraction Principle) Suppose $f : S^1 \to S^1$ is a circle homeomorphism has no periodic orbits and I is a subinterval of S^1. Then $\inf_{n\geq 1}\{|f^n(I)|\} = 0$, implies that I is a wandering interval of f.

Proof. Denote $I_n = f^n(int(I))$, and $\Sigma = \cup_{i\geq 0} f^i(I)$, then observe the following:

Case 1: $\Sigma = S^1$. By compactness, there're $I_{n_1}, I_{n_2}, \ldots, I_{n_k}$ a finite subcover of S^1. By Lebesgue Number Lemma, exists $\delta > 0$ such that every subset having diameter less than δ will be contained in one of $I_{n_1}, I_{n_2}, \ldots, I_{n_k}$. Since $\inf_{n\geq 1}\{|f^n(I)|\} = 0$, there must exist $I_l \subset S^1$, for some $l \in \mathbb{N}$ with $|I_l| < \delta$, and hence $I_j \in \{I_{n_i}\}_{i=1}^{k}$ such that $I_l \subset I_j$, so I is not a wandering interval. Wlog we suppose $l > j$, and note that $I_l = f^{l-j}(I_j)$, so f^{l-j} has a periodic point in the closure of I_j, a contradiction.

Case 2: $\Sigma \neq S^1$. If there is no component U *of* Σ such that some iterate of U intersects with U, then U and hence I are wandering intervals. If there is a component U *of* Σ such that $f^n(U) \cap U \neq \emptyset$ for some $n \geq 0$, then $f^n(U) \subseteq U$ and hence f^n has a periodic point in the closure of U. This is again a contradiction.

■

And now we shall begin the proof of **Theorem 2**.

Proof of **Theorem 2**. Let V be the upper bound of $Var(\log|Df|)$ on S^1, and suppose by contradiction that some $J \subset S^1$ is a wandering interval. Let q_n be the same as in **Lemma 1.16**, and $T = [f^{-q_n}(J), J]$ be the smallest interval containing J *and* $f^{-q_n}(J)$. By **Lemma 1.18**, $Dist(f^{q_n}, T) \leq V$, if we take the diameter of S^1 to be 1. Also by MVT, there exist $x \in J$ and $y \in f^{-q_n}(J)$ such that $|Df^{q_n}(x)| = \frac{|f^{q_n}(J)|}{|J|}$, $|Df^{q_n}(y)| = \frac{|J|}{|f^{-q_n}(J)|}$. These equations hold for all $n \in \mathbb{N}$.

Then for all n, $\frac{|J|^2}{|f^{q_n}(J)||f^{-q_n}(J)|} \leq \exp\big(Dist(f^{q_n}, T)\big) \leq \exp(V)$.

$|J|$ is a fixed number, so this means $\frac{1}{|f^{q_n}(J)||f^{-q_n}(J)|}$ is finite; but by contraction principle, this means $|f^{q_n}(J)|$ does not tend to 0, and hence by contradiction J is not a wandering interval. ■

Denjoy's theorem on topological conjugacy of circle diffeomorphisms inspired people to investigate further for regularities on diffeomorphisms so that such conjugacy holds; it is proven in late 20 century that diffeomorphisms satisfying the so-called *Zygmund* condition, $\sup_{x,t} |\frac{f(x+t)+f(x-t)-2f(x)}{t}| \leq B < \infty$, which clearly owns a similar structure of Dini's condition on convergence of Fourier sums.

Generally, C^1 and even C^∞ smoothness are not safe (see [9]) for conjugacy to an irrational rotation, and the non-existence of wandering sets appears to be the best tool of characterisation. A C^k diffeomorphism without periodic points for $k \geq 2$ can be proven to have no wandering sets since bounded variation of Df is guaranteed, hence these maps have S^1 as minimal set, we shall prove this in the next section. Furthermore, it can be proven that the only alternative is a topological Cantor set, hence the following question arises: for what type of Cantor set $K \subset S^1$ there exists a C^1 map with minimal set K. We will see some attempts to answer it from D.McDuff and later A.Portela, in Section III.

Another useful result follows from the properties of the set $S(\mathcal{J})$: consider the map $f: [0,1) \to [0,1)$ by

$$f(x) = \begin{cases} a + \frac{1-a}{b} x & for\ 0 \leq x < b \\ \frac{a}{1-b}(x-b) & for\ \ b \leq x < 1 \end{cases}$$

This piecewise affine map identifies with some circle homeomorphism, and can be further shown to have derivative of bounded variation depending on a and b. Therefore, by controlling the parameters of f so that $\rho(f)$ remains irrational, Denjoy's theorem asserts that f is conjugate to an irrational rotation map in $I([0,1])$. In this way, we can extend Denjoy's results to interval maps. (see [13])

Section II

Groups

When analysing the dynamics of a circle map, minimal sets (although we have not defined them so far) and wandering intervals are almost always studied together; one can easily check that existence of a non-empty wandering interval for some $f \in Homeo^{+}(S^1)$ implies that its minimal set cannot be dense in S^1, or equivalently, we say that the action of such f is not minimal or transitive on S^1. Hence minimal set is another useful tool to categorise circle maps. In addition, we can consider minimal sets for subgroups of circle homeomorphisms. Notice that if f is a circle homeomorphism without periodic points, then it generates an infinite cyclic group itself, so it does make sense to treat all minimal sets, whether they are minimal for a single function or a group of functions, as minimal sets of subgroups of $Homeo(S^1)$.

Also, while studying other types of dynamical systems, minimal set analysis is still a good starting point to examine the system, especially when the phase space is compact. A minimal set is strongly invariant when the phase space is compact Hausdorff, and if M is a minimal set of f, then $(M, f|_M)$ is necessarily a minimal system. Moreover, if a system (X, f) is minimal, which simply means X does not contain proper f-invariant sets, then it is analog of an ergodic measure in topological dynamics. One can find a very detailed research paper on minimal dynamical systems and the characterisation of minimal sets for low-dimensional systems by S.Kolyada and L'ubomir [15].

2.1 Definition and basic properties

Def. 2.1 Let $f: S^1 \to S^1$ be a circle homeomorphism, $A \subset S^1$ is invariant under f or f-invariant if $f(A) \subset A$.

A minimal set $K \subset S^1$ is a non-empty, compact subset invariant under f which has no proper non-empty invariant subset.

We will inherit the concept of wandering interval for the following:

$\Omega(f) \coloneqq \{x \in S^1 | x \text{ is not a wandering point of } f\}$, where x is a wandering point means that $\exists U \ni x$ an open interval such that $f^k(U) \cap U = \emptyset, \forall k \in \mathbb{Z}$.

Prop. 2.2 Let $f \in Homeo^+(S^1)$ without periodic points, then $\rho(f)$ is irrational and:

i. $\Omega(f) = \omega(x) = \alpha(x), \forall x \in S^1$

ii. $\Omega(f)$ is the minimal set of f

iii. Either $\Omega(f) = S^1$ or $\Omega(f)$ is a nowhere dense perfect set.

Proof.

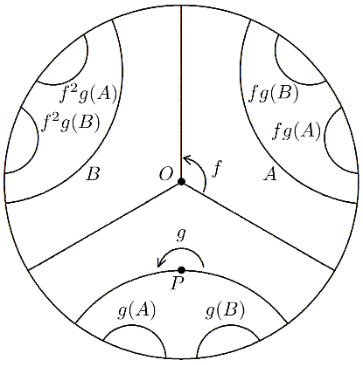
(i) Let $x \in S^1, and\ (a, b)$ be a component in $S^1 \backslash \omega(x)$, then $f^k((a, b))$ is also in $S^1 \backslash \omega(x)$ for all $k \in \mathbb{Z}$. Also, $\{f^j([a, b]) | j \in \mathbb{Z}\}$ must be pairwise disjoint because otherwise f would have periodic points, and hence (a, b) is a wandering interval of f. Therefore, $(S^1 \backslash \omega(x)) \subset (S^1 \backslash \Omega(f))$ which implies $\Omega(f) \subset \omega(x)$, and also $\omega(x) \subset \Omega(f)$ is clear by definition, so that $\Omega(f) = \omega(x)$, and this statement is indeed independent of x. $\Omega(f) = \alpha(x)$ can be proven in the same way for all $x \in S^1$.

(ii) $\Omega(f)$ is a closed set, and the unit circle is compact hence $\Omega(f)$ is compact. It is invariant exactly by the definition of limit sets, and any proper subset having the same properties will be empty.

(iii) Let $\partial\Omega(f)$ be the boundary of $\Omega(f)$, by closedness $\partial\Omega(f) \subset \Omega(f)$, and $f(\partial\Omega(f)) = \partial f(\Omega(f)) = \partial\Omega(f)$, by invariant property, hence either $\partial\Omega(f) = \emptyset$ or $\partial\Omega(f) = \Omega(f)$. In either case, the set is perfect by definition of $\omega(x)$. The first case implies that $\Omega(f) = S^1$ and in the second case, f is invertible and by definition of $\omega(x)$, no points will be isolated, so it is a perfect set whose interior is empty. Therefore it is a topological Cantor set. ■

The definition of $\Omega(f)$ ensures its uniqueness, and $\omega(x)$ is non-empty implies that any circle homeomorphism will attain a unique minimal set. Also, the proposition implies immediately the following: a circle homeomorphism with minimal Cantor set K is only semi-conjugate to an irrational rotation, since a homeomorphic conjugacy will preserve the topological structure of the orbits; equivalently, if there is a non-empty interval $J \subset S^1 \backslash \omega(x)$, it is a wandering interval of f.

The following is an example of a subgroup of $Homeo(S^1)$ attaining a Cantor minimal set.



Let f be the rotation by $2\pi/3$ angle, and P inside the unit circle with Euclidean distance greater than $2 - \sqrt{3}$ from the centre. Consider a rotation with centre P by angle π, let g be the real-analytic diffeomorphism on the circle corresponding to this rotation. Then, $< f, g >$ has a Cantor minimal set, demonstrated above by figure 3.

Figure 3 An example of a subgroup with minimal Cantor set from [16]

Remark This group corresponds with the so-called 'modular group' $PSL(2, \mathbb{Z})$, which has a finite group representation $< E, P: E^2 = P^3 = Id >$. But this correspondence is NOT a topological conjugacy, as on can prove that $< f, g >$
does not satisfy 'convergence property' in the next section, or verify that the limit set of $PSL(2, \mathbb{Z})$ is dense in the boundary.

2.2 Minimal sets of subgroups of circle homeomorphisms

In this subsection, we are going to investigate some group results which allow us to link subgroups of circle homeomorphisms which satisfy the 'convergence property' defined below, with Fuchsian subgroups of $PSL(2,\mathbb{R})$ or $con(1)$, a special type of subgroups frequently studied in the theory of hyperbolic geometry. This idea may not make so much sense at first glance, since these groups act on quite different domains, namely S^1 and the complex half-plane $\mathbb{H}^2$ (or the Poincaré disc $\mathbb{D}^2$) respectively. However, there is a powerful, difficult theorem (its proof will not be provided in this paper; check [11] theorem 2A, 2B and [14]) due to Casson-Jungreis, Gabai, Hinkkanen, and Tukia that says convergence property can further lead to topological conjugacy between circle homeomorphism subgroups and Fuchsian subgroups, therefore we can study them in pairs. It is a well-known fact that Fuchsian groups attain unique limit sets on the boundary of hyperbolic domains, and the structure of limit sets coincides with the three cases of minimal sets of circle homeomorphism subgroups. Therefore, since topological conjugacy preserves structures of limit sets/minimal sets, given a Fuchsian subgroup, we can apply techniques derived in hyperbolic geometry theory to obtain properties of the minimal set of its corresponding subgroup in $Homeo(S^1)$.

Prop. 2.5 and **Cor. 2.7** below confirm that convergence property does ensure topological conjugacy between circle homeomorphisms and transformations in $PSL(2,\mathbb{R})$, hence we are able to categorise circle homeomorphisms in the same way categorising Möbius transformations. **Theorem 5** concludes further that convergent subgroups are topologically conjugate to subgroups of $PSL(2,\mathbb{R})$, **Theorem 6 & 7** will then give all possible situations in which discrete convergence groups are conjugate to Fuchsian groups. Some results from hyperbolic geometry relating to limit sets of Fuchsian groups will be given as an evidence of resemblance between minimal sets of circle homeomorphism subgroups and limit sets of Fuchsian subgroups, and how we can use hyperbolic geometry theory to enrich analysis techniques for circle homeomorphism groups.

Let us first prove the general version of **Prop. 2.2** for minimal sets of subgroups of $Homeo(S^1)$. The orbit of some $x \in S^1$ under a subgroup $\Gamma \leq Homeo(S^1)$ means $\{f(x): f \in \Gamma\}$. It is often denoted as x^Γ in group theory settings, and a minimal set in this section means a subset $\Lambda \subset S^1$ such that $\Lambda^\Gamma = \Lambda$.

Theorem 4 If Γ is a subgroup of $Homeo(S^1)$, then one (and only one) of the following possibilities occurs:

i. there exists a finite orbit;

ii. all the orbits are dense in S^1, S^1 is invariant under Γ ;

iii. there exists a unique minimal invariant compact set which is homeomorphic to the Cantor set. Such set is contained in the set of accumulation points of every orbit.

Proof (original version can be found in [16] §2.1).

Obviously only one of the above items can happen. A finite orbit can occur when, for example, the group is generated by a single homeomorphism with finitely many fixed points or periodic points.

The family of non-empty closed invariant subsets in S^1 is partially ordered by inclusion, and since the unit circle is compact, it satisfies the non-empty intersection property, so a nested sequence of closed sets will attain a non-empty closed set. Zorn's lemma hence concludes the existence of a unique minimal invariant set Λ. Let Λ' be the set of accumulation points of Λ and $\partial\Lambda$ denote the boundary. By minimality of Λ, one of the three cases hold:

1. Λ' is empty, and Λ is a finite orbit.
2. $\partial\Lambda$ is empty, and hence $\Lambda = S^1$. All orbits under Γ are dense.
3. $\Lambda' = \partial\Lambda = \Lambda$, and Λ is homeomorphic to a Cantor set.

The first two cases are not difficult to verify: Λ is closed in S^1 hence sequentially compact, then if Λ is infinite, it must contain a limit point. And if $\partial\Lambda = \emptyset$, Λ is clopen hence has to be S^1.

For the last case, we can show that Λ is contained in the set of accumulation point of every orbit. Let $x \in S^1, y \in \Lambda$ be arbitrarily chosen; if $x \in \Lambda$, then since Λ is minimally invariant, there must exist a sequence $(g_n) \subset \Gamma$ such that $g_n(x) \to y$. If

$x \in S^1 \backslash \Lambda$, then Λ is closed implies its complement is open so we can select an open interval $(a, b) \ni x$ such that $a, b \in \Lambda$. The orbit of a under Γ is dense in Λ, and Λ does not contain isolated points, so there must exist $(g_n)_n$ such that $g_n(a)$ converges to y and $g_n((a, b))$ are pairwise disjoint so that f has no periodic points. Since $|g_n(I)| \to 0$, $g_n(x) \to y$, so for any $y \in \Lambda$ and $x \in S^1$, y belongs to the set of accumulation points of x^Γ, so $\Lambda' = \partial\Lambda = \Lambda$ indeed. ■

Remark If Γ has a finite orbit, then every orbit has exactly the same cardinality. Also conveniently, if we can prove that the minimal set Λ has an interior point, necessarily $\Lambda = S^1$.

Theorem 4 gives a simple categorisation of minimal sets of circle homeomorphism subgroups. Now we will see how certain subgroups of $Homeo(S^1)$ can be paired with subgroups of $PSL(2, \mathbb{R})$, especially, Fuchsian subgroups. The motivation comes from the fact that $PSL(2, \mathbb{R})$ whose elements are isometries on $\mathbb{H}^2$ which can be conjugated to the isometry group $con(1)$ on hyperbolic disk $\mathbb{D}^2$ via Cayley transformation ϕ, and non-elementary Fuchsian subgroups of $con(1)$ (hence Fuchsian in $PSL(2, \mathbb{R})$) have their limit sets either dense in the boundary of $\mathbb{D}^2$, i.e. $S^1 = \phi(\mathbb{R} \cup \{\infty\})$, or topologically equivalent to a Cantor set. There are various methods to analyse a Fuchsian group $G \leq PSL(2, \mathbb{R})$ and its limit set (which can be proven to be its minimal set, see **Prop. 2.9** below), e.g. finding the Dirichlet region/fundamental domains, apply Poincaré's theorem on its fundamental domains to obtain group representations, etc; by verifying topological conjugacy between a Fuchsian subgroup and a circle homeomorphism subgroup Γ, we will be allowed to inherit this rich toolbox from hyperbolic geometry for the analysis of Γ. Let us begin with the following definitions.

Definition 2.3 The group $PSL(2, \mathbb{R})$ refers to the set of Möbius transformations with a determinant condition, i.e.

$$PSL(2, \mathbb{R}) = \left\{ g(z) = \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R} \; and \; |ad - bc| = 1 \right\},$$

acting on the hyperbolic complex half-plane $\mathbb{H}^2$. It can be conjugated to an isometry group on Poincaré's disc model $\mathbb{D}^2$ which is bounded by S^1, via Cayley transformation $\phi(z) = \frac{z-i}{z+i}$. The hyperbolic metric on $\mathbb{D}^2$ is derived from $\frac{2|dz|}{1-|z|^2}, z \in \mathbb{D}^2$. Note also that each element in $PSL(2,\mathbb{R})$ induces a real-analytic diffeomorphism via ϕ on S^1, which identifies with the boundary of $\mathbb{D}^2$.

Moreover, we call a subgroup $\Gamma \leq PSL(2,\mathbb{R})$ *Fuchsian* if it is discrete with respect to the norm inherited from $\mathbb{R}^4$. The orientation preserving isometry group on $\mathbb{D}^2$ is given by:

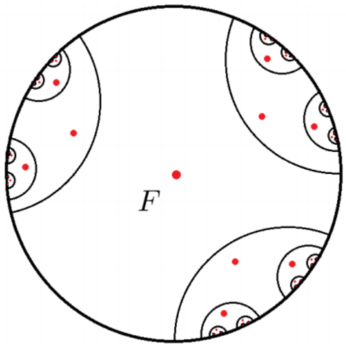


Figure 4 Cantor limit set of a Fuchsian subgroup, cited from [21]

$con^+(1) = \{g: \mathbb{D}^2 \to \mathbb{D}^2$: for some $a, b \in \mathbb{C}$ with $|a|^2 - |c|^2 = 1, g(z) = \frac{az+\bar{c}}{cz+\bar{a}}\}$, and $PSL(2,R) = \phi^{-1}\, con^+(1)\phi$, so any Fuchsian subgroup acts on $PSL(2,\mathbb{R})$ is a Fuchsian group acting on $\mathbb{D}^2$, therefore most statements made about Fuchsian groups apply to both models of hyperbolic domains. Also, all Fuchsian groups are necessarily countable.

The following notion initially introduced by Gehring and Martin is the important ingredient in order to conclude the topological conjugacy desired.

Definition 2.4 Let Γ be a group of circle homeomorphisms, we say that Γ is a *convergence group* or has the *convergence property* if, for each infinite sequence of distinct elements in Γ, we can find a subsequence $\{g_i\}$:

There are points x, y such that $g_i \to x$ and $g_i^{-1} \to y$ uniformly on $S^1 \backslash \{y\}$ and $S^1 \backslash \{x\}$ respectively.

Remark

1.The property is indeed quite strong, in the sense that only a small number of subgroups satisfy this condition. Also, it can be easily checked that this property is invariant under topological conjugacy.

2.There are multiple definitions of convergence groups on different domains, some of which require only pointwise convergence on compact sets (see [16]).

3. Every subgroup of $PSL(2,\mathbb{R})$ satisfies the convergence property.

The following proposition gives an insight of how this convergence property regulates the behaviour of elements.

Prop. 2.5 Let $g \neq id$ generate a convergence group on S^1, then g has at most 2 fixed points.

Proof (by Tukia). Suppose g fixes more than 2 points, then let $U = \{x \in S^1 | \ g(x) \neq x\}$. Clearly, U is a union of at least 3 disjoint open intervals, whose endpoints are fixed points of g, and each interval $I \subset U$ is fixed setwise by g. Then for each open interval I of U, $g|_I$ is topologically conjugate to the map $x \mapsto x + 1$ on $\mathbb{R}$ (it is also a parabolic element in $PSL(2,\mathbb{R})$). The order of g is then infinite since topological conjugacy preserves order, if $U \neq \emptyset$. This contradicts the assumption of convergence property, hence g can fix no more than two points in the unit circle. ■

Moreover, we are now able to borrow terminologies used to describe elements in $PSL(2,\mathbb{R})$ for circle homeomorphisms, and this is justified by the following corollary of **Prop. 2.5**, which is originally presented as a theorem in [11] 2A.

Def. 2.6 The trace of an element g in $PSL(2,\mathbb{R})$, or any Möbius transformation, say $g = \frac{az+b}{cz+d}$, is the sum of $a\ and\ d$. An element is so-called *hyperbolic* if $tr(g) > 4$, *parabolic* if $tr(g) = 4$ and *elliptic* if $tr(g) < 4$.

Remark An equivalent way to characterise elements is by the number of fixed points. The number of (distinct) fixed points in $R \cup \{\infty\}$ for a hyperbolic element is two, and one for a parabolic map. An elliptic element will have fixed points inside $\mathbb{H}^2$.

Cor. 2.7 If g generates a convergence group, then g can be conjugated by a homeomorphism to a Möbius transformation.

Proof.

Suppose g has 2 fixed points, say $x \; and \; y$, and suppose g is orientation-preserving, then g preserves the two components in $S^1 \backslash \{x, y\}$, and for each open interval $I \subset \{S^1 \backslash \{x, y\}\}$, $g|_I$ is conjugate to the map $x \mapsto x + 1$ on $\mathbb{R}$. Furthermore, by uniform convergence from convergence property, actions of g on $S^1 \backslash \{x, y\}$ is conjugate to the map $x \mapsto 2x$ on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ so that it has fixed points $\{0, \infty\}$ (in fact it can be $x \mapsto \alpha x$ for $\alpha > 1$) Thus g is conjugate to a hyperbolic Möbius transformation. If g is orientation-reversing, then it necessarily fixes 2 points, and it can be proven to be conjugate to either a Möbius reflection or a glide-reflection.

If g fixes one point, then it is conjugate to the parabolic map $x \mapsto x + 1$ on $\overline{\mathbb{R}}$.

Now suppose g has no fixed points. Let $A(x) = \{g^k(x) | \; k \in \mathbb{Z}\} = O_g(x) \cup O_{g^{-1}}(x)$.

If there's $x \in S^1$ such that $A(x)$ is finite, then g permutes the components of $S^1 \backslash A(x)$, $< g >$ is a group of cover transformations of S^1 and hence g is conjugate to an elliptic element. By **Fact 1.7** every such set is (asymptotic to) a periodic orbit.

Otherwise, suppose $A(x)$ is infinite for all points, then $< g >$ is necessarily infinite and the convergence property states that there is a sequence $g_i = g^{n_i}$ and $x, y \in S^1$ such that $g_i|_{S^1 \backslash \{x\}} \to y$ locally uniformly. However, for all $k \in \mathbb{Z}$, $g_i\left(g^k(z)\right) \to g^k(y)$ by continuity for every $z \in S \backslash \{g^{-k}(x)\}$ as $i \to \infty$. But as $A(y)$ is infinite by assumption, $g_i|_{S^1 \backslash \{x\}} y$ cannot be true, and the proof is complete. ■

And now, we can finally conclude the following.

Theorem 5. A group of circle homeomorphisms is topologically conjugate to a subgroup of $PSL(2, \mathbb{R})$ if and only if it satisfies the convergence property.

The proof of this statement will be out of the scope of this paper hence omitted, as it is lengthy and complicated in terms of preliminary knowledge in group theory and manifold theory. This wonderful result comes from that every subgroup of $PSL(2,\mathbb{R})$ satisfies the convergence property, and the action on the boundary of the hyperbolic plane under each element in $PSL(2,\mathbb{R})$ can be identified with a homeomorphism on S^1, therefore a convergent subgroup of circle maps should be intuitively 'equivalent' to a subgroup of $PSL(2,\mathbb{R})$.

Previous items justify categorising an orientation-preserving element of a discrete convergence group of circle homeomorphism as hyperbolic, parabolic and elliptic. In the case of an orientation-reversing homeomorphism, it can be seen as a reflection or a glide-reflection, where the latter means it is conjugate to a composition of a hyperbolic element with a reflection such that the hyperbolic fixed points are also fixed by the reflection.

The remaining part of this section mainly considers discrete convergence groups in $Homeo(S^1)$, as they are very likely to be topologically conjugate to a Fuchsian group. The name 'discrete' reflects that a convergence group G is a discrete subset of $Homeo(S^1)$ in the compact-open topology (assuming that Euclidean metric is taken on S^1), if and only if every sequence satisfies convergence property. The next definition relates to the limit sets.

Def. 2.8 Say group Γ acts *discontinuously* at a point $x \in S^1$ if exists a neighborhood $U \ni x$ such that $g(U) \cap U \neq \emptyset$ for finitely many $g \in \Gamma$.
Let $\Omega(\mathrm{G}) = \{x \in S^1 |\, G \text{ acts discontinuously on } x\}$. It is an open subset and we say G acts *properly discontinuously* on $\Omega(G)$ if for any compact $K \subset \Omega(G)$, $g(K) \cap K \neq \emptyset$ for finitely many g.

<u>Remark</u>

1. The idea of $\Omega(\Gamma)$ resembles that of wandering intervals, and we can almost immediately derive that:

$L(\Gamma) = \{z \in S^1 :$ there is a sequence $(g_i)\ in\ \Gamma$ such that $g_i(x) \to z\ for\ some\ x\}$ the limit set of G, is given by $L(G) = S^1 \backslash \Omega(G)$. Secondly, both $L(G)$ and $\Omega(G)$ are invariant under the action of G, and $L(G)$ is the minimal set of Γ.

2. In hyperbolic geometry, a subgroup of $PSL(2, \mathbb{R})$ is Fuchsian if and only if it acts properly discontinuously on $\mathbb{H}^2$. (see [18] section 2.2)

3. The limit set of a Fuchsian group G is given by the set of accumulation points in the orbit of any $z \in \mathbb{H}^2$ (or $\mathbb{D}^2$) under G.

Given a convergence group $\Gamma \leq Homeo(S^1)$, we know from **Theorem 4** that its minimal set has two different cases. Also, we can prove that a non-elementary Fuchsian group acts minimally invariant on its limit set, therefore we can equate limit sets with minimal sets from now on.

Prop. 2.9 The limit set of a non-elementary Fuchsian group acting on $\mathbb{D}^2$ is the smallest closed, invariant subset on S^1.

Proof. Suppose G is a non-elementary Fuchsian group. Let $E \subset S^1$ be a closed invariant set containing at least 2 points. Choose $u, v \in E$ and let C be the geodesic line segment in $\mathbb{D}^2$ joining u, v, and choose $w \in C$. Let $z \in L(G)$, and suppose the sequence $(g_n)_n$ sends w to z with respect to Euclidean metric. Using compactness of S^1, we can find subsequences such that $g_{n_k}(u) \to u'$and $g_{n_l}(v) \to v'$. If u', v' both distinct from z, then $g_n(w) \nrightarrow z$. Hence without loss of generality, assume $u' = z$. By invariant property, $z = \lim_{n_k \to \infty} g_{n_k}(u) \in E$, and by closedness $z \in E$.

Since G is non-elementary, $L(G)$ is infinite so any closed invariant set in S^1 with cardinality greater than 2 contains $L(G)$. Since G is non-elementary hence not generated by a single element, any invariant set cannot contain only 1 point. This completes the proof.■

Def. 2.10 Call G *elementary* if $L(G)$ at most two points, *of the first kind* if $L(G) = S^1$ and *of the second kind* if $L(G) \neq S^1$.

These terminologies are inherited from those of Fuchsian groups.

Now the last step is to confirm the topological conjugacy for discrete convergence subgroups so that we can apply hyperbolic geometry tools designed for analysing Fuchsian groups, to $G \leq \mathrm{Homeo}(S^1)$. The following two theorems list all possible situations, differentiated by the structure of minimal sets of G.

Theorem 6 If G is an elementary convergence group, then it is topologically conjugate to an elementary Fuchsian group.

Theorem 7 Let G be a discrete convergence group of S^1, then:

i. If G is non-elementary, either $L(G) = S^1$ or $L(G)$ is a perfect, closed nowhere dense subset of S^1. In particular, G is conjugate to a Fuchsian group.

ii. G acts properly discontinuously on $\Omega(G)$.

iii. If $x_1, x_2 \in \mathrm{L}(G)$, are distinct elements and U_i is a neighborhood of x_i, then exists hyperbolic $g \in G$ with one of P_g, N_g in U_1, U_2 separately.

iv. If $x \in \Omega(G)$ and $H_x = \{g: g \text{ is orientation preserving and } g(x) = x\}$, then either $H_x = \{id\}$ or H_x is infinitely cyclic.

Interestingly, the proof of **Theorem 6** is unexpectedly complicated even it seems that $L(G)$ has only 3 possibilities, which are 2 points, 1 point and an empty set; proof of **Theorem 7** case (i) follows almost immediately from **Theorem 4 and 5** with slight differences. Proof of the other three statements requires proposition 4.8 and 6.17 from [17]. Complete proofs will again be found in [11].

These theorems allow us to view discrete convergence subgroups of $Homeo(S^1)$ in the same way of studying Fuchsian groups, including their minimal sets and group presentation. Although these preceding theorems are not quite enough for asserting topological conjugacy between $\Gamma \leq Homeo(S^1)$ and a Fuchsian group in $PSL(2, \mathbb{R})$ –precise conditions are stated in [11] 6B – it is in fact only in very rare occasions (when there is a so-called semi-triangle subgroup of orientation-preserving homeomorphisms) the conjugacy is not homeomorphic.

The value of relating a convergence group of circle homeomorphisms to a Fuchsian group is that we have possibly a wider range of methods to characterise group elements in G $\leq Homeo(S^1)$ in terms of their (Möbius) conjugates, and the minimal set of Γ can be predicted to have the same structure of its conjugate Fuchsian group. For example, in the following case that a convergence group Γ is topologically conjugate to a Fuchsian group with bounded 'fundamental domain', the minimal set of Γ is known immediately.

Def. 2.11 Let $G \leq PSL(2, \mathbb{R})$ be Fuchsian, an open set $F \subset H^2$ is called a fundamental domain of G if $\bigcup_{g \in G} \overline{g(F)}$ and all $g, h \in G$ distinct, $g(F) \cap h(F) = \emptyset$. Fundamental domains are not unique, and always exist. There are multiple ways to find it, frequently done by finding the 'Dirichlet' region.

Theorem 8 A Fuchsian group with a bounded fundamental domain is of the first kind.
Proof. See [18] section 3.4.

Moreover, as long as the topological conjugacy is valid, we can borrow tools from hyperbolic geometry to obtain better description of the 'complexity' of minimal sets, for example exponent of convergence, which is, for a Fuchsian group G acting on the inside region of S^1 with hyperbolic metric, defined as

$$\delta(G) = inf\{s > 0\colon \sum_{z \in G(0)} (\frac{1-|z|}{1+|z|})\},$$

and in cases that G is finitely generated, $\delta(G)$ equals the Hausdorff dimension of the limit set.

<u>Remark</u>

There are equivalent ways to determine whether a group of circle homeomorphisms is a convergence group. Tukia, inspired by the action of Fuchsian sets on hyperbolic domains, gives the following solution.

Def. 2.13 Let T be the set of all triplets (u, v, w) in S^1, where u, v, w are distinct elements and in positive order. Then any homeomorphism g of S^1 defines a homeomorphism on T, where $g(u, v, w) = (g(u), g(v), g(w))$ if it is orientation-preserving, or $g(u, v, w) = (g(v), g(u), g(w))$ if orientation-reversing.

Theorem 9 Let Γ be a group of circle homeomorphisms, then Γ is a convergence group if an only if it acts properly discontinuously (in the same sense as **Def. 2.8**) on T.

Proof. See [11] 4A or [16]§1.3.3.

Section III

Cantor minimal sets of diffeomorphisms

Now we will come back to minimal sets of a single circle homeomorphism. Every $f \in Homeo^+(S^1)$ without periodic points by **Prop. 2.2** (or **Theorem 4**, since we can treat its dynamics as group actions of $< f >$ on S^1), has a unique minimal set. It is also a known fact that the group of circle homeomorphisms acts transitively on the set of Cantor subsets of S^1, which means for every $K \subset S^1$topologically Cantor set, there is $f \in Homeo^+(S^1)$ such that K is the minimal set of f; and for any irrational number λ, there is a C^1 diffeomorphism with rotation number λ but with a minimal Cantor set (see theorem 2.3 in [2]) but is not true that any Cantor subset K is minimal set for a C^1 diffeomorphism, therefore M. Herman proposed the question that for which Cantor subsets K there exists a C^1 diffeomorphism of S^1 having minimal set K, and we call such K C^1 minimal. The question can be partially answered in different directions: if K is C^1 minimal, it is 'locally' C^1 minimal; if K is C^1 minimal, the lengths of connected components in its complement need to satisfy certain limiting conditions; and if K satisfies the p-separation condition introduced by Portela, K is not C^1 minimal. A non-trivial consequence we will introduce here by D.McDuff in [8] §1.4 asserts that the usual ternary Cantor set is not C^1-minimal.

3.1 McDuff's Condition

First, McDuff in [8] §3 has given a simple case of what makes a set C^1 minimal. The idea is that we can construct for the (non-empty) intersection of a Cantor C^1minimal set K with an open arc, which is itself still Cantor, a C^1 diffeomorphism with it being the unique minimal set, by 'collapsing' the semi-conjugacy to a smaller circle that is identified with the open arc.

Prop. 3.1 If K is minimal for the C^1-diffeomorphism f, and let J be a non-empty arc of the form $(x, f^k x)$, for $x \in K^c$. Then $J \cap K$ is also C^1 minimal.

Proof. K is the minimal Cantor set of f implies that f is semi-conjugate to an irrational rotation by **Prop. 2.2** and **theorem 1**, so there is a continuous, monotone, surjective map $h: S^1 \to S^1$ such that $h \circ f = R_\alpha \circ h$, where $\alpha = \rho(f)$. Since K is minimally invariant under f, $h(K) = S^1$; moreover, one can actually prove that h maps each connected component in K^c to a single point, and $h(K^c)$ is countable. Also, h is continuous, and each open interval $I \subset K^c$, has its endpoints in K, we have that h is one-one restricted to $\{K \backslash \{x\}: x \in \bar{I}, for\ some\ I \subset K^c\}$. (McDuff refers to this set as 'interior points' of K. [8] §2)

Then, given that this semi-conjugacy is invariant under any rotation on the left, we can choose $h(x) = 0$, so that $h(f^k x) = k\alpha\ (mod\ 1)$, $|h(J)| = k\alpha\ (mod\ 1)$. Let T be the circle with length $\beta = k\alpha\ (mod\ 1)$ obtained from S^1 by identifying all points in $S^1 \backslash h(J)$ as a single point, and $\pi: S^1 \to T$ be the projection. The map $\tilde{h}: S^1 \to T$, given by $\tilde{h} = \pi \circ h$ is surjective as it is composition of two surjective functions, and one-one on the 'interior' of $J \cap K$.

Choose m so that $\gamma = m\alpha\ (mod\ 1)$, $\frac{\gamma}{\beta} < 1$ is irrational. Let $\tau: T \to T$ be the rotation by γ, it should have no periodic point and the countable set $D = \tilde{h}(K^c)$ is invariant under τ: T is identified with the positively oriented arc $[0, \beta) \subset S^1$, then τ is translation by γ on $[0, \beta - \gamma)$ and translation by $-(\beta - \gamma)$ on $[\beta - \gamma, \beta)$. τ can be lifted to a C^1 diffeomorphism g of S^1 by $\tilde{h}$. As we show now: if $I = (a, b) \ni x$ is a component of K^c, then we can construct such diffeomorphism g in the following way: $g|_{[b, f^{k-m}(a)]} = f^m$, $g|_{[f^{k-m}(b), f^k(a)]} = f^{m-k}$ so that they will be well-defined for T, and extent them on the rest of S^1 by any C^1 diffeomorphisms from $f^{k-m}(\bar{I})$ to $[f^k(a), b]$ and $[f^k(a), b]$ to $f^m(\bar{I})$ which that coincides with previous components of g near the end of these intervals. This diffeomorphism is semi-conjugate to τ by $\tilde{h}$, an irrational rotation on the smaller circle T obtaining from collapsing the points in $S^1 \backslash h(J)$; as m is chosen so that τ has minimal set T, and $\tilde{h}$ maps the interior points of $J \cap K$ injectively onto $T \backslash \tilde{h}(K^c)$, which is dense in T (also the set $\tilde{h}(K^c)$ is a countable union of singletons), then the minimal set of g is $J \cap K$. ■

<u>Remark</u> Rotation number of g is γ/β.

This diffeomorphism is actually not unique, because we can choose different m satisfying conditions stated in the proof of **3.1**. Also, since $K = \omega(f)$, every $x \in K$ is contained in some open arc $A = (f^l(y), f^m(y))$ for some $y \in K^c$, and this open set can be made arbitrarily small, and by **Prop. 3.1**, $A \cap K$ is C^1-minimal. McDuff refers to this property as 'locally minimal'.

It is equally hard to answer to 'what is not C^1minimal'. From sections 1.3 and 2.1, the existence of a minimal Cantor set is due to the existence of a wandering interval, and the length of its images under f, f^{-1} tend to 0. This is in fact a vague statement, for example in the construction of the famous ternary Cantor set K on [0,1], the length of open intervals being inductively removed obviously tends to 0, but if $\{\lambda_i\}$ are numbers in decreasing order such that they equal to the length of the open intervals in the complement of K, $limsup_n \, \lambda_n/\lambda_{n+1} = 3$, and one can verify that if a subset of S^1 identifies with this ternary Cantor set, it cannot be C^1 minimal. Very often, a Cantor set failing to be C^1 minimal results from the fact that the derivative of a diffeomorphism changes little on a small set. We will revisit this statement soon.

Suppose K is a Cantor subset of S^1, let $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ be the lengths of connected components in K^c; we may also call the set $\{\lambda_i\}_{i \in \mathbb{N}}$ the *spectrum* of K. Let $J_j = [\alpha_j, \beta_j], j \geq 1$ be disjoint closed subintervals of [0,1], arranged in decreasing order so λ_j, λ_{j+1} either belong to the same J_i, or $\lambda_j \in J_i, \lambda_{j+1} \in J_{i+1}$. In other words, $\{\lambda_j\}_{j \geq 1}$ is covered by $\{J_i\}_{i \geq 1}$ ($\alpha_j = \beta_j$ is allowed, and the cover is simply a union of singletons), and the following holds:

Lemma 3.2 If K is C^1 minimal, then the 'gap ratios', $\frac{\alpha_j}{\beta_{j+1}}$, are bounded.

Proof. It suffices to show that $\frac{\lambda_j}{\lambda_{j+1}}$ are bounded, as $\alpha_{i+1} \le \lambda_{j+1} \le \beta_{i+1} < \alpha_i \le \lambda_j$ for $\lambda_j \ne \lambda_{j+1}$, and $\lambda_j \in J_i, \lambda_{j+1} \in J_{i+1}$.

Suppose K is minimal for some $\mathcal{C}^1$-diffeomorphism f, and choose $c > 0$ (since f is orientation preserving) such that $Df(x) \ge c, \forall x \in S^1$. Then by MVT, the length of each interval I (endpoint s of I are in K) in K^c under f, $\ell(fI) \ge c\ell(I)$. Since f has no periodic point (existence of a minimal Cantor set implies that its rotation number is irrational) and K^c is invariant under f, for any connected component $I \subset S^1$, $f^i(I) \cap f^j(I) = \emptyset$ if $i \ne j$, hence $\lim_{k\to\infty} \ell(f^k I) = 0$.

Then for any $i \ge 1$, there is $N_i \in \mathbb{N}$ such that $\ell(f^{N_i} I) > \lambda_{i+1}$, and for all $k > N_i$, $\ell(f^k I) \le \lambda_{i+1}$. Hence let $I' = f^{N_i} I$, we have $\frac{\lambda_i}{\lambda_{i+1}} \le \frac{\ell(I')}{\ell(fI')} \le c^{-1}$, and this is true for all $i \ge 1$. ■

This lemma implies that the lower and upper bounds of two adjacent $\mathcal{J}_j's$ are quite close for large j, leading to the following proposition, which is the most important result in this subsection, that says when K cannot be $\mathcal{C}^1$ minimal; a generalised version by A. Portela will be proven in §3.2.

Prop. 3.3 Let $\lambda_i, \alpha_i, \beta_i$ be the same as above, then if for each $N > 0$, there is $\eta(N) > 0$ such that

$$\frac{\alpha_{j+n-1}}{\beta_{j+n}} \ge \big(1 + \eta(N)\big)\frac{\beta_j}{\alpha_j}, \text{ for } -N \le n \le N \text{ and all } j > N \ (\star),$$

K is NOT $\mathcal{C}^1$ minimal.

Remark In simpler language, this means every 'adjacent gap' ratio is greater than the 'cover' ratio $\frac{\beta_j}{\alpha_j}$.

The intuition is that the derivative of a $\mathcal{C}^1$ diffeomorphism changes very little if the interval I in K^c is small (which gives Lemma 3.5 below), and ($\star$) enlarges the gap between λ_i and λ_{i+1}, hence K satisfying ($\star$) cannot be $\mathcal{C}^1$ minimal.

We will seek help from some more definitions and lemmas to establish the proof.

Def. 3.4 In order to keep consistency with the original work by McDuff, we let $\ell(I)$ be the length of a connected component I in K^c. The *depth* of a connected component I in K^c, $d(I)$, is the integer j such that $\ell(I) \in J_j = [\alpha_j, \beta_j]$.

Lemma 3.5 Suppose K is C^1 minimal for f, and $(\star)$ is satisfied, then K can be covered by finitely many disjoint open arcs $A_1, A_2, \dots, A_r$ such that: if for some $1 \leq i \leq r, I, I' \subset A_i \cap K^c$, then $d(I) \leq d(I')$ implies $d(fI) \leq d(fI')$.

Proof (slightly modified from the original version in [8]).

For $N = 1$, there is $\eta = \eta(1) > 0$ such that $\frac{\alpha_j}{\beta_{j+1}} \geq (1+\eta) \cdot \frac{\beta_j}{\alpha_j} \geq 1 + \eta$, for all j. From **Lemma 3.2**, α_j/β_{j+1} can be bounded by some $L > 1$, so:

$$\frac{\beta_j}{\alpha_j} \leq \frac{\alpha_j}{\beta_{j+1}} \leq L \text{ for all } j.$$

Then $(1+\eta)^M \leq \frac{\alpha_j}{\beta_{j+1}} \cdot \frac{\alpha_{j+1}}{\beta_{j+2}} \dots \frac{\alpha_{j+M-1}}{\beta_{j+M}} = \frac{\alpha_j}{\beta_{j+M}} \cdot \frac{\alpha_{j+1}}{\beta_{j+1}} \dots \frac{\alpha_{j+M-1}}{\beta_{j+M-1}} \leq \frac{\alpha_j}{\beta_{j+M}}$ for all $j > 1$, where $\frac{\alpha_j}{\beta_{j+M}} = \frac{\alpha_j}{\beta_{j+1}} \cdot \frac{\beta_{j+1}}{\alpha_{j+1}} \cdot \frac{\alpha_{j+1}}{\beta_{j+2}} \cdot \frac{\beta_{j+2}}{\alpha_{j+2}} \dots \frac{\beta_{j+M-1}}{\alpha_{j+M-1}} \cdot \frac{\alpha_{j+M-1}}{\beta_{j+M}} \leq L^{2M}$,

combining the inequalities, for all $j > 1$, and all $M > 0$, we have

$$(1+\eta)^M \leq \frac{\alpha_j}{\beta_{j+M}} \leq L^{2M} \quad (i).$$

Since $(\star)$ says $\eta > 0$, and f is C^1 diffeomorphic on a compact set, it is possible to choose M large enough so that

$$\frac{\alpha_j}{\beta_{j+M}} \geq (1+\eta)^M > \sup\{Df(x), Df^{-1}(x) : x \in S^1\}, \text{ for all } j.$$

Now suppose that $I \subset K^c$ is an open interval, and $\ell(I) \in J_j = [\alpha_j, \beta_j]$ for some j.

By MVT, there is $\xi \in I$, with $\frac{\ell(I)}{\ell(fI)} = \frac{1}{|f'(\xi)|} = |f^{-1}(f(\xi))| < \alpha_j/\beta_{j+M}$, which implies that $1 \leq \frac{\ell(I)}{\alpha_j} < \frac{\ell(fI)}{\beta_{j+M}}$ hence $\beta_{j+M} < \ell(fI)$.

Then $$|d(I) - d(fI)| \leq M \quad (ii),$$

for all connected component $I \subset K^c$ with length smaller than λ_1. The same statement can be proven similarly for $f^{-1}I$ and $d(f^{-1}I)$.

Choose $\delta < \frac{\eta(M)}{L^{2M}}$. K is compact, therefore we can cover K with finitely many open arcs $\{A_i\}_{i=1}^{r}$ whose lengths are small enough such that:

(1) $|Df(x) - Df(y)| < \frac{\delta}{2}$, $\forall x, y \in A_i, 1 \leq i \leq r$. It gives that if two components $I, I' \subset K^c \cap A_i$, $\left|\frac{\ell(fI)}{\ell(I)} - \frac{\ell(fI')}{\ell(I')}\right| < \delta$ by MVT again.

(2) If $I \subset K^c \cap A_i$ for some $1 \leq i \leq s$, then $d(I) \geq M$, which means $\ell(I) \leq \beta_M$.

Suppose by contradiction that there are I, I' connected components in K^c, belong to some A_i with $d(I) \leq d(I')$ but $d(fI) > d(fI')$. By assumption (2), $M \leq d(I) = j$, and we can assume that $j + n = d(fI) > d(fI')$, for some $1 \leq n \underbrace{\leq}_{by\ (ii)} M$.

Then, $\ell(I') \leq \beta_j, \ell(fI') \geq \alpha_{j+n-1} > \beta_{j+n}$, which gives

$$\frac{\ell(fI')}{\ell(I')} - \frac{\ell(fI)}{\ell(I)} \geq \frac{\alpha_{j+n-1}}{\beta_j} - \frac{\beta_{j+n}}{\alpha_j} = \left(\frac{\alpha_{j+n-1}}{\beta_{j+n}} - \frac{\beta_j}{\alpha_j}\right)\frac{\beta_{j+n}}{\beta_j} \quad (iii).$$

Apply condition $(\star)$ to the right hand side of (iii), we get

$$\frac{\ell(fI')}{\ell(I')} - \frac{\ell(fI)}{\ell(I)} \geq \eta(M) \cdot \frac{\beta_j}{\alpha_j} \cdot \frac{\beta_{j+n}}{\beta_j} = \eta(M) \cdot \frac{\beta_{j+n}}{\alpha_j},$$

by (i), $\frac{\beta_{j+n}}{\alpha_j} \geq \frac{\beta_{j+M}}{\alpha_j} \geq \frac{1}{L^{2M}}\ for\ all\ j > 1$, so LHS of $(iii) \geq \frac{\eta(M)}{L^{2M}} > \delta$, which contradicts with assumption (1) above. ■

The immediate consequence is the lemma below, which will prove **prop. 3.3**.

Lemma 3.6 Suppose K is C^1 minimal for f and satisfies $(\star)$, then there is some L such that for all $0 < \varepsilon \leq L$, there exists an open subset $U \subset S^1$, with $U \cap K$ non-empty, and for all $I \subset K^c \cap U$, $\ell(f^k I) < L_0\varepsilon\ \ \forall k \geq 0$, $L_0 = \sup\{\beta_j/\alpha_j\}$.

Proof (an adapted version of the original proof in [8]§4.9).

The existence of L_0 is implied by ($\star$) and **Lemma 3.2**.

Let $\{A_i\}_{i=1}^r$ be the set of open arcs satisfying **Lemma 3.5**, choose L small enough so that all open intervals in K^c with length smaller than L_0L are strictly contained in some A_i; such is possible because all endpoints of open intervals in K^c are elements of K.

Let $\varepsilon > 0$ be given and define the set

$\mathcal{J} = \{I \subset K^c | I \text{ is an open interval with } \ell(f^kI) < \varepsilon \leq L_0\varepsilon \text{ for all } k \geq 0\}$.

Since K is minimal Cantor set if and only if f has a wandering interval I', and by disjointness $\lim_{|k|\to\infty} \ell(f^kI') = 0$, there exists N such that for all $k \geq N, \ell(f^kI') < \varepsilon$, then put $I = f^NI', I \in \mathcal{J}$, hence $\mathcal{J}$ is necessarily non-empty. In particular, one can select a wandering $I_{max} \in \mathcal{J}$ of maximal length with $\ell(I_{max}) < \varepsilon$, because there are finitely many intervals $J \in \mathcal{J}$ with $\ell(J) \in [\frac{\varepsilon}{2}, \varepsilon]$. Then, by assumption, there is A_i strictly containing I_{max}, and let U be a connected open subset of A_i strictly containing the closure of I_{max}, such that if an open interval I' of K^c is contained in U, it satisfies $\ell(I') \leq \ell(I_{max}) < \varepsilon$, and $d(I') \geq d(I_{max})$.

U strictly contains the closure of I_{max} ensures that $U \cap K \neq \emptyset$. Now use induction on the following statement:

$U(k)$: for all open interval $I' \subset K^c \cap U, d(f^kI') \geq d(f^kI_{max})$. This means for some $j \in \mathbb{N}$, $\frac{\ell(f^kI')}{\ell(f^kI_{max})} \leq \frac{\beta_j}{\alpha_j} \leq L_0$.

$U(0)$ clearly holds, as any I' contained in U by assumption gives $\ell(I') \leq \ell(I_{max}) < \varepsilon L_0$.

Pick an arbitrary open interval $I' \subset U \cap K^c$. As $I_{max}, I' \subset U \subset A_i, \ell(I') \leq \ell(I_{max}) \Rightarrow d(I') \geq d(I_{max}) \Rightarrow d(fI') \geq d(fI_{max})$, therefore $U(1)$ holds and $\ell(fI') \leq L_0\ell(fI_{max}) < L_0\varepsilon$. So if $I' \subset U, \ell(fI') < L_0\varepsilon$.

Assume the statement holds up to $U(k-1)$, any $I' \subset U$ satisfies $\ell(f^{k-1}I') < L_0\ell(I_{max}) < L_0\varepsilon$ implies that f^kI' is in some A_j. The same holds for f^kI_{max}, and since f is a diffeomorphism, $f^k(U)$ is connected which means all connected components in U are in the same A_j, so $f(U) \subset A_j$ for some $1 \leq j \leq r$. Then by **Lemma 3.5**, $d(f^{k-1}I') \geq d(f^{k-1}I_{max}) \Rightarrow d(f^kI') \geq d(f^kI_{max})$ for all I' contained in U, hence $U(k)$ holds for all $k \geq 0$ and the proof is complete. ■

Remark L_0 is non-trivially bounded below by 1, as we do allow the cover of the spectrum of K to be a union of singletons.

Proof of **Prop. 3.3**.

In order to prove that K satisfying $(\star)$ is not $\mathcal{C}^1$, McDuff finds a handy contradiction to **Lemma 3.6**:

Suppose K satisfies $(\star)$ and is $\mathcal{C}^1$ minimal for some diffeomorphism f, let $\{A_i\}$ be the same as above, then pick $\varepsilon > 0$ and the open set U, so that we can choose some component $I = (x, y)$ in the complement of K such that $\ell(I) > L_0\varepsilon > \ell(I_{max})$, so I is not in U.

$U \cap K \neq \emptyset$, select $k \in U \cap K$, so there is an open ball $B(k, \mu) \subset U$ for some $\mu > 0$.

$K = \alpha(x) = \alpha(y)$, so exists $k > 0$ such that $f^{-k}(x), f^{-k}(y) \in B(k, \mu)$, and by connectedness $f^{-k}I \subset B(k, \mu) \subset U$. Now put $I' = f^{-k}I$, observe that $I = f^kI' > L_0\varepsilon$, a contradiction. ■

3.2 Portela's *p*-Separation Condition.

In a fairly recent work by A.Portela, a generalisation **Prop. 3.3** is given with.

Def. 3.7 A Cantor subset K of S^1 satisfies *p-separation* condition, denoted by $\star_p$, if a covering of the spectrum of K, $\{\mathcal{J}_j\} = \bigcup_j [\alpha_j, \beta_j]$ satisfies:

Exists $p \geq 0$ that for all $N > 0$, there is $\eta(N) > 0$ such that

$$\frac{\alpha_{j+n-1}}{\beta_{j+n+p}} \geq \big(1 + \eta(N)\big) \cdot \frac{\beta_j}{\alpha_{j+p}},$$

for any integer $|n| \leq N$ and all $j > N$.

Remark In [3] the author originally used the term “sufficiently large” for condition on j.

Theorem 10 If K is a Cantor subset of S^1 satisfies the condition $(\star_p)$, it is not $\mathcal{C}^1$ minimal.

We are going to review the proof provided by Portela in 52[3], with a few changes in notations and techniques in **Lemma 3.10** to keep consistency with McDuff’s results; clearly, the three lemmas below needed for proving **Theorem 10** are generalisations or modifications based on previous ones used in §3.1.

Lemma 3.8 If K is $\mathcal{C}^1$ minimal and satisfies $(\star_p)$ for $\{\mathcal{J}_j\}_{j\geq 1}$, then β_j/α_j is also bounded.

Proof. Taking $N = n = 1$, we get

$$\frac{\alpha_j}{\beta_{j+p+1}} \geq \big(1+\eta(1)\big) \cdot \beta_j/\alpha_{j+p} \text{ for } j > 1,$$

Then
$$\frac{\beta_j}{\alpha_j} \leq \frac{\alpha_{j+p}}{\beta_{j+p+1}} \cdot \frac{1}{1+\eta(1)}$$

By **Lemma 3.2** above, λ_i/λ_{i+1} is bounded for all i, therefore the ratios $\frac{\alpha_{j+p}}{\beta_{j+p+1}}$ are bounded, and so will be β_j/α_j.■

Lemma 3.9 If K is $\mathcal{C}^1$ minimal for f and satisfies $(\star_p)$, then $|d(I) - d(fI)|$ is bounded for every component $I \subset K^c$.

Proof. Pick an arbitrary component I of K^c, suppose $d(fI) > d(I)$, and observe that for $j > N = 1$, by $(\star_p)$, when $n = 0$,

$$\frac{\alpha_{j-1}}{\beta_{j+p}} \geq \big(1+\eta(1)\big) \cdot \frac{\beta_j}{\alpha_{j+p}}, \text{ therefore } \frac{\alpha_{j-1}}{\beta_j} \geq \big(1+\eta(1)\big) \cdot \frac{\beta_{j+p}}{\alpha_{j+p}} \geq \big(1+\eta(1)\big),$$

and we have:

$$\left(1+\eta(1)\right)^{d(fI)-d(I)} \leq \frac{\alpha_{d(I)}}{\beta_{d(I)+1}} \cdot \frac{\alpha_{d(I)+1}}{\beta_{d(I)+2}} \cdot \ldots \cdot \frac{\alpha_{d(fI)-1}}{\beta_{d(fI)}} \leq \frac{\alpha_{d(I)}}{\beta_{d(fI)}} \leq \frac{\ell(I)}{\ell(fI)} \leq \frac{1}{m},$$

where $m := \inf_{x \in S^1} Df(x)$.

If $d(I) \geq d(fI)$ then exchange the role of $d(I)$ and $d(fI)$ in the last inequality, and we can conclude $\left(1+\eta(1)\right)^{d(I)-d(fI)} \leq \frac{\ell(fI)}{\ell(I)} \leq M$, where $M := \sup_{x \in S^1} f'(x)$. Since $1 + \eta(1) > 1$, $|d(I) - d(fI)|$ must be bounded in both cases. ■

Lemma 3.10 Suppose the Cantor set K is C^1 minimal for f and satisfies $(\star_p)$ for some $p \geq 0$, then there is a cover of K consists of finitely many open intervals $\{T_i\}_{i=1}^r$ such that: if I, J are components in K^c and contained in some T_i, then

$$d(I) - d(J) \leq p \text{ implies } d(fI) - d(fJ) \leq p.$$

Proof.

Let N_0 be the upper bound of $\{|d(I) - d(fI)| : I \subset K^c\}$ given by **Lemma 3.9**.

Similar to the proof of **Lemma 3.5**, we choose $\delta > 0$ such that $\frac{\delta}{m} < \eta(N_0)/2$. Then we can find an open cover $\{T_i\}_{i=1}^r$ of K consists of sufficiently small open arcs, such that:

1.If $x, y \in T_i$, then $|f'(x) - f'(y)| < \delta$, which further implies that

$$\frac{f'(x)}{f'(y)} < \frac{f'(y)+\delta}{f'(y)} < 1 + \frac{\delta}{m} < 1 + \eta(N_0)/2,$$

2. If a component I of K^c is contained in some T_i, $d(I) \geq N_0 + 1$.

We now show this $\{T_i\}_{i=1}^r$ is the cover desired. Suppose components I, J of K^c contained one T_i satisfy $d(I) - d(J) \leq p$, but $d(fI) - d(fJ) > p$, or equivalently $d(fJ) + p < d(fI)$. Then $\frac{\ell(fJ)}{\ell(fI)} \geq \frac{\alpha_{d(fJ)}}{\beta_{d(fI)}} \geq \frac{\alpha_{d(fJ)}}{\beta_{d(fJ)+p+1}}$, let $n = |d(J) - d(fI)| \leq N_0$ (this inequality is guaranteed by **Lemma 3.9**), we can apply $(\star_p)$ condition to obtain:

$$\frac{\ell(fJ)}{\ell(fI)} \geq \frac{\alpha_{d(J)+n}}{\beta_{d(J)+n+p+1}} \geq \big(1+\eta(N_0)\big) \cdot \frac{\beta_{d(J)}}{\alpha_{d(J)+p}}.$$

By MVT, there are $\xi \in J$ and $\zeta \in I$ such that

$$\frac{\ell(fJ)}{\ell(fI)} = \frac{\ell(J)}{\ell(I)} \cdot \frac{f'(\xi)}{f'(\zeta)} \leq \frac{\beta_{d(J)}}{\alpha_{d(I)}}\delta \leq \frac{\beta_{d(J)}}{\alpha_{d(J)+p}}\delta < (1+\frac{\eta(N_0)}{2})\frac{\beta_{d(J)}}{\alpha_{d(J)+p}},$$

which is a contradiction to the first assumption of $\{T_i\}_{i=1}^{r}$. ■

*Proof of **Theorem 10***. Suppose K satisfies $(\star_p)$ for some $p \in \mathbb{N}$.

Since $T = \bigcup_{i=1}^{r} T_i$ is an open cover of K, the complement T^c is closed and compact, hence can be covered by finite number of components of K^c, $\{L_1, L_2, \dots, L_s\}$. Let $I \subset K^c$, $\ell(f^k I) \to 0$ as $|k| \to \infty$, so we can find j_0 such that for all $j > j_0$, we have that

$$d(f^j I) > p + \max\{d(L_1), \dots, d(L_s)\}$$

As $\{L_i\}$ are disjoint components, there exists $1 \leq i_0 \leq r$ such that $f^{j_0} I = (a_{j_0}, b_{j_0}) \subset T_{i_0}$. Choose $c_{j_0} \in K$ so that $\ell\big((c_{j_0}, a_{j_0})\big) < \ell\big(f^{j_0} I\big)$. If J is a component of K^c contained in (c_{j_0}, a_{j_0}), then

$$\left|d\big(f^{j_0} I\big) - d(J)\right| \leq p, \text{which implies } \left|d\big(f^{j_0+1} I\big) - d(fJ)\right| \leq p.$$

By choice of j_0, $d(fJ) \geq -p + d\big(f^{j_0+1} I\big) > \max\{d(L_1), \dots, d(L_s)\}$, so $f(J)$ is not in T^c. Similar to the proof of **Lemma 3.5**, we would have $f(f^{j_0} I)$ is completely contained in some T_{i_1}, and inductively $f^k\big(f^{j_0} I\big) \subset T_{i_k}$, any $J \subset (c_{j_0}, a_{j_0})$ satisfies $f^n(J) \neq L_i, \forall i = 1,2, \dots, s$.

But since the α-limit set of any L_i is K, for each $i \in \{1,..s\}$ necessarily we would find infinitely many $k > 0$ such that $f^{-k} L_i \subset (c_{j_0}, a_{j_0})$, so by contradiction, K cannot be $\mathcal{C}^1$minimal. ■

<u>Remark</u>

1. McDuff's condition $(\star)$ above is a special case, $(\star_{p=0})$.

2. In the proof above, (c_{j_0}, a_{j_0}) is not necessarily a connected component in the complement of K.

3. Let K be $\mathcal{C}^1$ minimal and $\{\lambda_i\}_{i\geq 1}$ is fixed; then we can choose for each $j \geq 1$ with $\alpha_j = \beta_j = \lambda'_j$, where $\lambda'_j = \min_{i\leq j} \lambda_i$, then the contrapositive of **Prop. 3.3** with $N = 1$ gives that if K is $\mathcal{C}^1$ minimal then λ_i/λ_{i+1} has 1 as a non-trivial limit. Portela generalises this argument in [3] § 1.4 that if for some $\varepsilon > 0$, the covering $\{[\alpha_j, \beta_j]\}_j$ satisfies $\frac{\alpha_j}{\beta_{j+1}} = 1 + \varepsilon$ as well as $\frac{\beta_j}{\alpha_j} \equiv k$ then using **Theorem 10** we get that K is not $\mathcal{C}^1$ minimal.

3.3 Two Examples

The first example is the ternary Cantor set of the unit circle, which is not $\mathcal{C}^1$ minimal. The covering of its spectrum, $\{J_j\}_{j\in\mathbb{N}}$, can be chosen to be exactly the countable set of singletons $\left\{\frac{2\pi}{3^j}\right\}_{j\in\mathbb{N}}$, so let $\eta(N) = 2\ \forall N$, we have for all j, $\frac{\alpha_j}{\beta_{j+1}} \geq (1+2)\beta_j/\alpha_j$, hence it satisfies $(\star)$ and by **Prop. 3.3** it is not $\mathcal{C}^1$ minimal.

The following example is given by Portela as a demonstration of p-separation condition with $p = 1$ in [3]§3.

Let $\gamma < 3 < \gamma^{3/2}$, and define for $n \geq 1$, $A(n) = \{\frac{\gamma^{\frac{j}{2n}}}{3^{4n+2}} : |j| \leq n\}$.

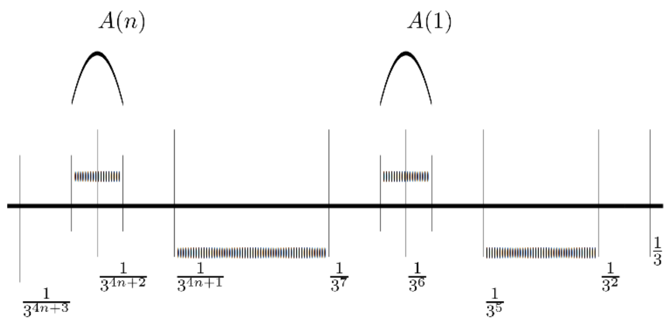


Figure 5 Distribution of elements in $A(n)$ on [0,1]. See [3] for the original version.

The sum $S(n) = \sum_{\alpha \in A(n)} \alpha = \sum_{j=-n}^{n} \frac{\gamma^{\frac{j}{2n}}}{3^{4n+2}} \leq \gamma^{1/2} \sum_{j=-n}^{n} \frac{1}{3^{4n+2}} \leq \frac{\gamma^{1/2}}{3^{2n}}$, hence $\sum S(n) < \infty$.

Let $B = \left\{\frac{1}{3^i} : i \geq 1\right\} \cup \bigcup_{n \in \mathbb{N}} A(n)$, and $\sum_{\beta \in B} \beta \leq 1/2 + \sum S(n) =: \mu < \infty$.

Define $C = \{\frac{2\pi x}{\mu} : x \in B\}$, and the family of open intervals $\{(e^{ia_j}, e^{ib_j})\}$ is constructed via the following:

Set B is a countable union of countably infinite sets, hence countable, and there is a bijection $m: \mathbb{N} \to C$.

$$a_0 = 0, b_0 = m(0)$$

Now take an irrational rotation R_θ of angle θ, and pick arbitrary $x \in S^1$,

$$a_i = b_0 + \sum_{R_\theta^k(x) \in \left(x, R_\theta^j(x)\right)} m(k) \ , \qquad b_j = m(j)$$

Let $K = S^1 \backslash (\bigcup_j (e^{ia_j}, e^{ib_j}))$, then the spectrum of K, i.e. the lengths of open intervals in its complement is the set C. Portela did not explicitly prove this set is a topological Cantor set, thus we are doing so now.

Prop. 3.11 K is a perfect, nowhere dense closed set.

Proof.

Claim: $\{(e^{ia_j}, e^{ib_j})\}_j$ is a disjoint family of open sets.

Proof of claim: take $j, l \in \mathbb{N}, j \neq l$, and without loss of generality, suppose $e^{ia_j} < e^{ia_l}$ (it means the smallest interval between these points, $\left(e^{ia_j}, e^{ia_l}\right)$, is positively oriented) , then it suffices to show $e^{ia_l} > e^{ib_j}$.

$$\ell\left(\left(e^{ia_j}, e^{ia_l}\right)\right) = \sum_{R_\theta^k \in (x, R_\theta^l)} m(k) - \sum_{R_\theta^k \in (x, R_\theta^j)} m(k)$$

Since $e^{ia_j} < e^{ia_l}$, $\left(x, R_\theta^j\right) \subset (x, R_\theta^l)$, so $\ell\left(\left(e^{ia_j}, e^{ia_l}\right)\right) = \sum_{R_\theta^k \in [R_\theta^j, R_\theta^l)} m(k)$, which is strictly greater than $m(j)$, therefore $\left(e^{ia_j}, e^{ib_j}\right) \cap \left(e^{ia_l}, e^{ib_l}\right) = \emptyset$, the claim holds.

The sum of lengths of components in K^c is 2π, hence K^c is dense in $S^1 \Rightarrow K$ is nowhere dense.

K is clearly closed. If $x \in K$ is isolated, it is a common endpoint of some intervals in K^c, i.e. for some $j, l \in \mathbb{N}, e^{ib_j} = e^{ia_l} = x \Rightarrow a_l - a_j = m(j)$. But then as before, $\sum_{R_\theta^k \in [R_\theta^j, R_\theta^l)} m(k) > m(j)$, this is impossible. Therefore K does not contain isolated points, i.e. a perfect set, and nowhere dense.

Prop. 3.12 K satisfies ($\star_1$) condition.

Proof. Elements in C are of the form

$$\omega_i = \frac{2\pi}{\mu 3^i},\ \omega_{i,j} = \frac{2\pi\gamma^{\frac{j}{2i}}}{\mu 3^{4i+2}}, i \in \mathbb{N}, |j| \leq i,$$

Therefore $\frac{2\pi\gamma^{-1/2}}{\mu 3^{4i+2}} \leq \ \omega_{i,j} \leq \frac{2\pi\gamma^{1/2}}{\mu 3^{4i+2}}$ since $f(x) = \gamma^x$ is monotone increasing.

To construct $\{\mathcal{J}_i\}_{i \geq 1}$ for C, consider :

$$\begin{cases} \alpha_i = \dfrac{2\pi\gamma^{-1/2}}{\mu 3^i}, \beta_i = \dfrac{2\pi\gamma^{1/2}}{\mu 3^i}, & when\ i = 4k+2, k > 0 \\ \alpha_i = \beta_i = \dfrac{2\pi}{\mu 3^i}, & otherwise \end{cases}$$

Remember that $\gamma < 3 < \gamma^{3/2}$, so for all integer n, we have

$$\frac{\alpha_{j+n-1}}{\beta_{j+n+1}} \geq \frac{2\pi}{\mu 3^{j+n-1}} / \frac{2\pi\gamma^{\frac{1}{2}}}{\mu 3^{j+n+1}} = 9\gamma^{-\frac{1}{2}} > \gamma^{3/2}.$$

And $\frac{\beta_j}{\alpha_{j+1}} \leq \frac{2\pi\gamma^{\frac{1}{2}}}{\mu 3^j} / \frac{2\pi}{\mu 3^{j+1}} \leq 3\gamma^{\frac{1}{2}}$, so for all $N > 0$, pick $\eta(N) \equiv \gamma^{-\frac{1}{2}} - 1 > 0$, then for all $|n| \leq N$, and $j > N$,

$$\frac{\alpha_{j+n-1}}{\beta_{j+n+1}} > \gamma^{3/2} > 3 \geq \gamma^{-\frac{1}{2}} \frac{\beta_j}{\alpha_{j+1}} = \big(1+\eta(N)\big) \frac{\beta_j}{\alpha_{j+1}}.$$

Remark

This Cantor set in fact does not satisfy McDuff's ($\star_0$) condition, the proof is not too hard and will be omitted here.

Conclusion

This last section is a review of the mathematical results introduced in previous parts of this paper. Some further questions will be proposed as potential topics for future research extension.

In Section I, classical results from early the 20th century concerning the conjugacy of circle homeomorphisms are presented. We introduced the concept of 'rotation number', explained how it can be derived via combinatorics based on the dynamics of the circle map following steps instructed in [2]. Poincaré used this number as an indicator of semi-conjugacy to an irrational rotation, and Denjoy improved this to a topological conjugacy by requiring the map to be $\mathcal{C}^1$whose derivative is of bounded variation. The concept of a wandering interval appears in deriving Denjoy's theorem and is used in later discussion of minimal sets.

In Section II, using the concept of wandering intervals, it is proven first that if no finite orbit exists, the unique minimal set of a subgroup in $Homeo(S^1)$ is either dense in the circle, or topologically equivalent of a Cantor set. Then, assuming convergence property, circle maps are characterised as hyperbolic, parabolic and elliptic by the number of fixed points, and two separate theorems in [11] together conclude that most discrete convergence groups of circle homeomorphisms are topologically conjugate to Fuchsian subgroups of $PSL(2,\mathbb{R})$, hence their limit sets are topologically equivalent, so that the toolbox for analysing minimal sets of circle map groups is enriched by the theory of hyperbolic geometry.

Section III focuses on the study of minimal Cantor sets, inspired by M. Herman's question "for which Cantor subset K of S^1 there exists a corresponding $\mathcal{C}^1$ diffeomorphism with minimal set K". We examined responses from D. McDuff, who concluded both that a $\mathcal{C}^1$ minimal Cantor set is necessarily locally $\mathcal{C}^1$ minimal, and if the spectrum of K is covered by some disjoint closed intervals with particular limiting conditions, K is not C^1 minimal. The latter proposition was generalised by A. Portela's p-separation condition, and one particular Cantor set was explicitly constructed as an example satisfying the p-separation condition and therefore not C^1 minimal.

There remain several unresolved questions related to the study of C^1 minimal Cantor sets that are worth further research.

Let $\mathcal{C}(r) = \{K : K$ is Cantor minimal set for some C^r diffeomorphism $f\}$. As a corollary of **Theorem 2** by Denjoy in section I, the derivative Df of any C^2 diffeomorphism f is of bounded variation hence topologically conjugate to an irrational rotation, so $\mathcal{C}(r) = \emptyset$ for any $r \geq 2$; and we know every Cantor subset of S^1 is in $\mathcal{C}(0)$. The results demonstrated in previous subsections give restrictions on elements in $\mathcal{C}(1)$, with requirements on the sequence $\{\lambda_i\}$ of lengths of components in the complement. There are two types of questions remaining, the first one, raised by McDuff herself in [8] asks if $\lambda_n/\lambda_{n+1} \to 1$ is necessary and sufficient for C^1 minimal, provided that all C^1 minimal Cantor sets known or constructed indeed all satisfy the condition, and any Cantor subset K with $\lambda_n/\lambda_{n+1} \nrightarrow 1$ is certainly not C^1 minimal. One may ask another question seemingly less interesting but still worth thinking, that whether it is possible to construct for each $p \in \mathbb{N}$ a Cantor set satisfying $(\star_p)$ condition, and a corresponding $f \in Homeo^+(S^1)$. Further, can we extend these results to interval maps with a few discontinuities, similar to the idea of $S(\mathcal{J})$ in section I? For example, for which Cantor subset K of $[0,1]$ there is a piecewise C^1 diffeomorphism with K as minimal set?

More generally, for $\mathcal{C}(1+\varepsilon)$, $\forall \varepsilon \in [0,1)$, it is almost always easier to conjecture what type of Cantor set is not in $\mathcal{C}(1+\varepsilon)$, which usually done by writing out the explicit construction process of the Cantor sets; for example in [19] A.N.Kercheval has shown an 'affine' Cantor set cannot be in $\mathcal{C}(1)$, and A.Portela in another paper has shown in [20] that a special type, namely the quasi-regular interval Cantor set of S^1, with non-zero regularity, cannot be an element in $\mathcal{C}(1+\varepsilon)$ for all $\varepsilon > 0$.

This type of results generates more open questions such as:

1. Can we measure how 'far' is a Cantor subset of the unit circle from being a member of $\mathcal{C}(\alpha)$, for some $\alpha \in (0,2)$? For example, one can use the following distortion-like formula described in [19]

$$\mathcal{N}(f) = \max_{j=1,2,..k} \sup_{x,y \in I_j} log \frac{f'(x)}{f'(y)},$$

for some function f defined on disjoint compact intervals $I_1, \dots, I_k$, such that $f[I_j] = L = cl(\bigcup_{j=1}^{k} I_j)$. Is there a universal measurement on Cantor sets to determine where they belong to $\mathcal{C}(1)$?

2. Is there anything we can assert for $\mathcal{C}(\alpha)$, for $0 < \alpha < 2, \alpha \neq 1$?

These questions are potential topics for further investigations.

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